A1.

a) Let $A \in \mathbb{R}^{n \times n}$ be rank 1. Denote j-th column of A by A_j
without loss of generality assume $A_1 \neq 0$

Any column A_j can be represented as $A_j = \alpha_j A_1$

Let us take $u = A_1$ and $v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$, then $A = uv^T$

b) $B_1 = B_1$ · $B_2 = 2 \cdot B_1$; $B_3 = 5 B_1$.

$\Downarrow$

B is rank 1 matrix (Any column B_j can be represented as $B_j = \alpha_j B_1$)

Then $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 5$.

Using a) we get $u = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

c) Using definition if x is eigenvector then
$Bx = \lambda x$.

Th If $B = uv^T$ Then u is eigenvector with eigenvalue $\langle u, v \rangle$

Proof $Bu = u \underbrace{v^T u}_{\text{Scalar}} = u \langle u, v \rangle$

As far as matrix is rank 1 two other eigenvalues are equal to 0.

To find eigenvectors we need to find some basis of $Ker(B)$.

Hint: if $y \in Ker(B)$ then $\langle y, u \rangle = 0$.

Let's take vectors $u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $u_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$\langle u_2, u \rangle = 0$; $\langle u_3, u \rangle = 0$; $\langle u_2, u_3 \rangle$ – it's ortogonal basis.

d) Check proof for b)

e) Let's prove the hint first.

$$\underbrace{\begin{bmatrix} I & 0 \\ v^T & 1 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} I + uv^T & u \\ 0 & 1 \end{bmatrix}}_{A_2} = \begin{bmatrix} I + uv^T & u \\ v^T + v^T u v^T & v^T u + 1 \end{bmatrix}$$

$$\begin{bmatrix} I + uv^T & u \\ v^T + v^T u v^T & v^T u + 1 \end{bmatrix} \underbrace{\begin{bmatrix} I & 0 \\ -v^T & 1 \end{bmatrix}}_{A_3} = \underbrace{\begin{bmatrix} I + \cancel{uv^T} - \cancel{uv^T} & u \\ \cancel{v^T} + \cancel{\underbrace{v^T u}_{\text{scalars}} v^T} - \cancel{v^T \underbrace{v^T u}_{\text{scalars}}} - \cancel{v^T} & v^T u + 1 \end{bmatrix}}_{B}$$

$A = A_1 \cdot A_2 \cdot A_3$

$A = B \implies \det(A) = \det(B)$

$\det(A) = \det(A_1) \cdot \det(A_2) \cdot \det(A_3) = \det(I + uv^T)$

$\det(B) = 1 + v^T u$

f) Let us check via multiplication.

$$\left(I - \frac{uv^T}{1 + v^T u}\right)\left(I + uv^T\right) =$$

$$= I + uv^T - \frac{uv^T}{1 + v^T u} - \frac{u\overbrace{v^T u}^{\text{scalar}} v^T}{1 + v^T u} =$$

$$= I + \cancel{uv^T} - \left(\frac{\cancel{(1 + v^T u)} uv^T}{\cancel{1 + v^T u}}\right) = I$$

B.1.

a) $A^T A x = A^T b$ $\qquad A^T A = \tilde{A} \quad A^T b = \tilde{b}$

$\tilde{A} x = \tilde{b}$ $\qquad$ $\uparrow$ symmetric

If $\tilde{A}$ is full rank solution is unique $x = \tilde{A}^{-1}\tilde{b} = (A^T A)^{-1} A^T b$

First way

Let $\tilde{A}$ be not full rank.

Then there exists vector x such that $\tilde{A}x = 0$

$\tilde{A}x = A^T A x$

A is full rank $\Rightarrow Ax \neq 0$

A^T is full rank as far as A is full rank $\Rightarrow A^T y \neq 0$ $\quad \forall y \neq 0$

If $y = Ax$ then $\tilde{A}x = A^T A x = A^T y \neq 0$

This contradiction concludes a proof.

Second way

In case of $V \in \mathbb{R}^{n \times d}$ $\quad \boxed{V^* = V^H = V^T = V^{-1}}$

Let $A = USV^*$ - SVD of matrix A

Matrix S is of form $\begin{array}{cc} & d \\ n & \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \end{array}$ where Σ is diagonal

$A^T = VS^TU^*$ - SVD of matrix A^T

$A^T A = VS^T \underbrace{\cancel{U^*U}}_{I} SV^* = VS^TSV^* = V\Sigma^2 V^*$

$$\boxed{S^T \begin{bmatrix} \Sigma & 0 \end{bmatrix} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} S = \Sigma^2 \qquad \boxed{S^TS = \Sigma^2}}$$

A is full rank

$\Downarrow$

Σ is full rank

$\Downarrow$

Σ^2 is full rank

$\Downarrow$

$A^T A = V\Sigma^2 V^*$ is full rank.

b) Using notation of second way we need to express $x = (A^T A)^{-1} A^T b$

$$A^T A = V \Sigma^2 V^*$$

$$(A^T A)^{-1} = (V \Sigma^2 V^*)^{-1} = V \Sigma^{-2} V^*$$

$$(A^T A)^{-1} A^T = V \Sigma^{-2} \underbrace{\cancel{V^* V}}_{Id} S^T U^* = V \Sigma^{-2} S^T U^* = V \bar{S} U^*$$

$$\bar{S} = \Sigma^{-2} S^T$$

$$S^T \quad [\Sigma \;\; 0]$$

$$\Sigma^{-2} [\Sigma^{-1} \;\; 0]$$

$$\bar{S}$$

C1. General theory.

Matrix R

By definition

$$\boxed{x_i = \sum_{j \in P_i} \frac{x_j}{n_j} = \sum R_{ij} x_j} \quad \text{so} \quad \boxed{R_{ji} \triangleq \begin{cases} 0 & \text{if } j \notin P_i \\ \frac{1}{n_j} & \text{o.w.} \end{cases}}$$

Let us prove that R is column stochastic.

→ all elements R_{ij} are nonnegative!

→ $\sum_{i=1}^{n} R_{ij} = \sum_{Q_j} \frac{1}{n_j} = 1$ where Q_j - pages j points toward.

$$Rx = x \iff (R - I)x = 0$$

As far as R is column stochastic any column of $(R-I)$ sums to 0.

It implies that $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ such that $x_1 = x_2 \ldots = x_n$

is a solution of $(R-I)^T x = 0$.

So x is in kernel of $(R-I)^T$.

$\Downarrow$

$(R-I)^T$ is not full rank

$\Downarrow$

$\det(R-I)^T = 0$

$\Downarrow$

$\det(R-I) = 0$

$\Downarrow$

$R-I$ is not full rank

$\Downarrow$

There exists $y \neq 0$ in kernel of $R-I$

$\Downarrow$

$(R-I)x = 0$ has nontrivial solution.

this solution is not unique as far as λx is also solution $(R-I)\lambda x = \lambda(R-I)x = \lambda 0 = \lambda$.

The solution of $Rx = x$ is a-priory an eigenvector with eigenvalue equal to 1

Let us use $\|\cdot\|_{1,1}$.
from b) it follows that $\|R\|_{1,1} = 1$.

$$\|R^n\|_{1,1} \leq \underbrace{\|R\|_{1,1} \cdot \ldots \cdot \|R\|_{1,1}}_{p \text{ times}} = 1$$

$\rho(R^n) \leq \|R^n\|_{1,1} = 1 \Rightarrow (R^n)_n$ stays bounded

As far as 1 is eigenvalue of R spectral radius cannot be less than 1 $\begin{cases} \rho(R) \geq 1 \\ \rho(R) \leq 1 \end{cases} \Rightarrow \rho(R) = 1$
but also it cannot be greater

for e) see the Th. 8.3.1 from „Matrix Analysis"