Proximal Method with Contractions for Smooth Convex Optimization

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Grenoble
September 23, 2019

Plan of the Talk

1. Proximal Method with Contractions

2. Application to Second-Order Methods

3. Numerical Example

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$$f^* = \min_{x \in \mathbb{R}^n} f(x)$$

Proximal Method:

$$x_{k+1} = \operatorname*{argmin}_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2a_{k+1}} \|y - x_k\|^2 \right\}.$$

[Rockafellar, 1976]

- If f is convex, the objective of the subproblem $h_{k+1}(y) = f(y) + \frac{1}{2a_{k+1}} \|y - x_k\|^2$ is strongly convex.
- Let f has Lipschitz gradient with constant L_1. Gradient Method needs $\tilde{O}\big(a_{k+1} L_1\big)$ iterations to minimize h_{k+1}.
- It is enough to use for x_{k+1} an inexact minimizer of h_{k+1}.

[Solodov-Svaiter, 2001; Schmidt-Roux-Bach, 2011; Salzo-Villa, 2012]

Set $a_{k+1} = \frac{1}{L_1}$. Then $f(\bar{x}_k) - f^* \leq \frac{L_1 \|x_0 - x^*\|^2}{2k}$.

Accelerated Proximal Method

Denote $A_k \stackrel{\mathrm{def}}{=} \sum_{i=1}^k a_i$. Two sequences: $\{x_k\}_{k\geq 0}$, and $\{v_k\}_{k\geq 0}$.

Initialization: $v_0 = x_0$.

Iterations, $k \geq 0$:

1. Put $y_{k+1} = \frac{a_{k+1}v_k + A_k x_k}{A_{k+1}}$.
2. Compute $x_{k+1} = \operatorname*{argmin}_{y\in\mathbb{R}^n}\Big\{ f(y) + \frac{A_{k+1}}{2a_{k+1}^2}\|y - y_{k+1}\|^2 \Big\}$.
3. Put $v_{k+1} = x_{k+1} + \frac{A_k}{a_{k+1}}(x_{k+1} - x_k)$.

Set $\frac{a_{k+1}^2}{A_{k+1}} = \frac{1}{L_1}$. Then

$$f(x_k) - f^* \;\leq\; \frac{8L_1\|x_0 - x^*\|^2}{3(k+1)^2}.$$

[Nesterov, 1983; Güler, 1992; Lin-Mairal-Harchaoui, 2015]

- *A Universal Catalyst for First-Order Optimization.*
- What about **Second-Order** Optimization?

Iterations, $k \geq 0$:

1. Compute $v_{k+1} = \operatorname*{argmin}\limits_{y \in \mathbb{R}^n} \left\{ A_{k+1} f\left(\frac{a_{k+1} y + A_k x_k}{A_{k+1}}\right) + \beta_d(v_k; y) \right\}$.

2. Put $x_{k+1} = \frac{a_{k+1} v_{k+1} + A_k x_k}{A_{k+1}}$.

$\beta_d(x; y)$ is Bregman Divergence.

Basic setup: $\beta_d(x; y) = \frac{1}{2}\|y - x\|^2$. Then

$$A_{k+1} f\left(\frac{a_{k+1} y + A_k x_k}{A_{k+1}}\right) + \frac{1}{2}\|y - v_k\|^2 \;=\; A_{k+1}\left(f(\tilde{y}) + \frac{A_{k+1}}{2a_{k+1}^2}\|\tilde{y} - y_{k+1}\|^2 \right),$$

where $\tilde{y} \equiv \frac{a_{k+1} y + A_k x_k}{A_{k+1}}$ and $y_{k+1} \equiv \frac{a_{k+1} v_k + A_k x_k}{A_{k+1}}$.

- The same iteration as in *Accelerated Proximal Method*.
- Generalization to arbitrary prox-function $d(\cdot)$.

Let $d(y)$ be a convex differentiable function. Denote **Bregman Divergence** of $d(\cdot)$, centered at x as

$$\beta_d(x;y) \stackrel{\mathrm{def}}{=} d(y) - d(x) - \langle \nabla d(x), y - x \rangle \geq 0.$$

- Mirror Descent [Nemirovski-Yudin, 1979]
- Gradient Methods with Relative Smoothness [Lu-Freund-Nesterov, 2016; Bauschke-Bolte-Teboulle, 2016]

Consider regularization of convex $g(\cdot)$ by Bregman Divergence:

$$h(y) \equiv g(y) + \beta_d(v;y).$$

Main Lemma. $T = \operatorname*{argmin}_{y \in \mathbb{R}^n} h(y)$. Then

$$h(y) \geq h(T) + \beta_d(T;y).$$

Proximal Method with Contractions: the Main Idea

We want, for all $y \in \mathbb{R}^n$:

$$\beta_d(x_0; y) + A_k f(y) \quad \geq \quad \beta_d(v_k; y) + A_k f(x_k). \qquad \textbf{(\$)}$$

How to propagate it to $k+1$? Denote $a_{k+1} \stackrel{\mathrm{def}}{=} A_{k+1} - A_k > 0$.

$$\begin{aligned}
\beta_d(x_0; y) + A_{k+1} f(y) \quad &\equiv \quad \beta_d(x_0; y) + A_k f(y) + a_{k+1} f(y) \\
&\stackrel{(\$)}{\geq} \quad \beta_d(v_k; y) + A_k f(x_k) + a_{k+1} f(y) \\
&\geq \quad \beta_d(v_k; y) + A_{k+1} f\left(\tfrac{a_{k+1} y + A_k x_k}{A_{k+1}}\right) \equiv h_{k+1}(y).
\end{aligned}$$

Let $\boldsymbol{v_{k+1} = \operatorname*{argmin}_{y \in \mathbb{R}^n} h_{k+1}(y)}$. Then, by the Main Lemma,

$$\begin{aligned}
h_{k+1}(y) \quad &\geq \quad h_{k+1}(v_{k+1}) + \beta_d(v_{k+1}; y) \\
&\geq \quad A_{k+1} f\underbrace{\left(\frac{a_{k+1} v_{k+1} + A_k x_k}{A_{k+1}}\right)}_{\equiv x_{k+1}} + \beta_d(v_{k+1}; y).
\end{aligned}$$

Iterations, $k \geq 0$:

1. Compute $v_{k+1} = \underset{y \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ A_{k+1} f\left(\frac{a_{k+1} y + A_k x_k}{A_{k+1}}\right) + \beta_d(v_k; y) \right\}$.

2. Put $x_{k+1} = \frac{a_{k+1} v_{k+1} + A_k x_k}{A_{k+1}}$.

Rate of convergence:

$$f(x_k) - f^* \quad \leq \quad \frac{\beta_d(x_0; x^*)}{A_k}.$$

Questions:

- How to choose A_k? Prox-function $d(\cdot)$?
- How to compute v_{k+1}?

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$$h^* = \min_{x \in \mathbb{R}^n} h(x)$$

h is convex, with Lipschitz continuous Hessian:

$$\|\nabla^2 h(x) - \nabla^2 h(y)\| \leq L_2 \|x - y\|.$$

Model of the objective

$$\Omega_M(x; y) \stackrel{\mathrm{def}}{=} h(x) + \langle \nabla h(x), y - x \rangle + \tfrac{1}{2} \langle \nabla^2 h(x)(y - x), y - x \rangle + \tfrac{M}{6} \|y - x\|^3$$

Iterations:

$$\boxed{z_{t+1} := \operatorname*{argmin}_{y \in \mathbb{R}^n} \Omega_M(z_t; y), \quad t \geq 0.}$$

Newton method with Cubic regularization [Nesterov-Polyak, 2006]

- Global convergence

$$h(z_t) - h^* \leq O\left(\tfrac{L_2 R^3}{t^2}\right).$$

Apply **Cubic Newton** to compute the Proximal Step:

$$h_{k+1}(y) \;\; \equiv \;\; A_{k+1} f\left(\frac{a_{k+1}y + A_k x_k}{A_{k+1}}\right) + \beta_d(v_k; y) \;\; \to \;\; \min_{y \in \mathbb{R}^n}$$

- Pick $d(x) = \frac{1}{3}\|x - x_0\|^3$.
- Uniformly convex objective: $\beta_h(x; y) \geq \frac{1}{6}\|y - x\|^3$. Linear rate of convergence for Cubic Newton:

$$h(z_t) - h^* \;\; \leq \;\; O\left(\exp\left(-\frac{t}{\sqrt{L_2}}\right)(h(z_0) - h^*)\right).$$

- Let v_{k+1} be inexact Proximal Step: $\|\nabla h_{k+1}(v_{k+1})\|_* \leq \delta_{k+1}$.

Theorem

$$f(x_k) - f^* \;\leq\; \frac{\left(3^{-2/3}\|x_0 - x^*\|^2 + 6^{1/3}\sum_{i=1}^{k}\delta_i\right)^{3/2}}{A_k}$$

- $O\left(\sqrt{L_2(h_{k+1})}\log\frac{1}{\delta_{k+1}}\right)$ iterations of Cubic Newton for step k.

Contracted objective: $g_{k+1}(y) \;\equiv\; A_{k+1} f\left(\frac{a_{k+1}y + A_k x_k}{A_{k+1}}\right)$.

Derivatives

1. $Dg_{k+1}(y) = a_{k+1} Df\left(\frac{a_{k+1}y + A_k x_k}{A_{k+1}}\right)$,
2. $D^2 g_{k+1}(y) = \frac{a_{k+1}^2}{A_{k+1}} D^2 f\left(\frac{a_{k+1}y + A_k x_k}{A_{k+1}}\right)$,
3. $D^3 g_{k+1}(y) = \frac{a_{k+1}^3}{A_{k+1}^2} D^3 f\left(\frac{a_{k+1}y + A_k x_k}{A_{k+1}}\right)$,

   $\ldots$

Notice: $D^{p+1} f \;\preceq L_p(f) \;\Rightarrow\; D^{p+1} g_{k+1} \;\preceq\; \frac{a_{k+1}^{p+1}}{A_{k+1}^p} L_p(f)$. Therefore,

if we have $\boxed{\frac{a_{k+1}^{p+1}}{A_{k+1}^p} \leq \frac{1}{L_p(f)}}$ then $L_p(g_{k+1}) \leq 1$.

- For Cubic Newton ($p = 2$) set $A_k = \frac{k^3}{L_2(f)}$. We obtain accelerated rate of convergence: $O\left(\frac{1}{k^3}\right)$.

Basic Method

$p = 1$: Gradient Method.

$p = 2$: Newton method with Cubic regularization.

$p = 3$: Third order methods (admits effective implementation) [Grapiglia-Nesterov, 2019].

$\ldots$

- Prox-function: $d(x) = \frac{1}{p+1}\|x - x_0\|^{p+1}$. Set $A_k = \frac{k^{p+1}}{L_p(f)}$.
- Let $\delta_k = \frac{c}{k^2}$.

Theorem

$$f(x_k) - f^* \quad \leq \quad O\left(\frac{L_p(f)\|x_0 - x^*\|^{p+1}}{k^{p+1}}\right).$$

- $O\left(\log \frac{1}{\delta_k}\right)$ steps of *Basic Method* every iteration.

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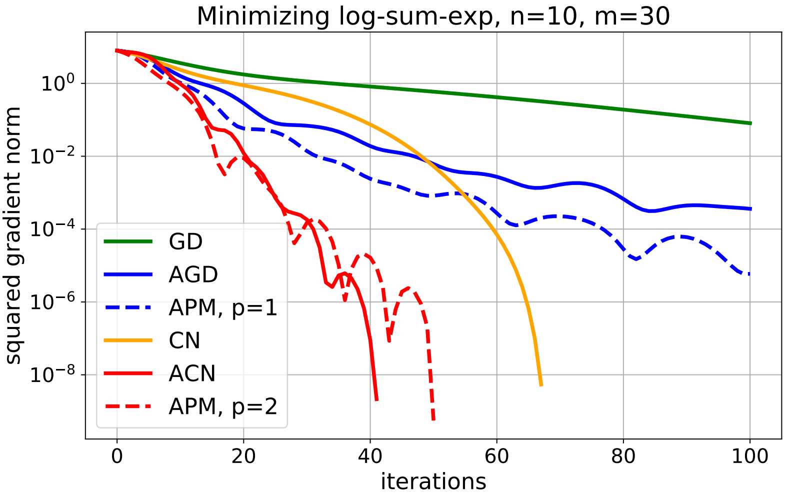
$$\min_{x \in \mathbb{R}^n} f(x) \;=\; \log\left(\sum_{i=1}^{m} e^{\langle a_i, x\rangle}\right).$$

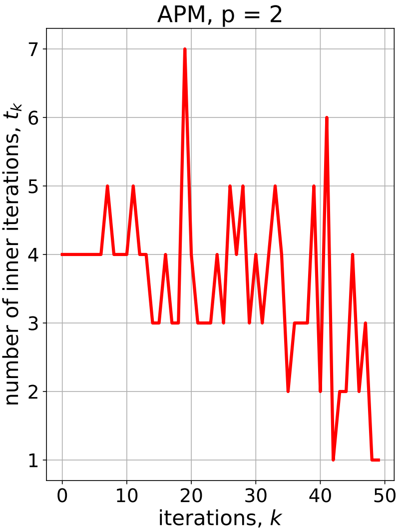
- $a_1, \dots, a_m \in \mathbb{R}^n$ — given data.
- Denote $B \equiv \sum\limits_{i=1}^{m} a_i a_i^T \succeq 0$, and use $\|x\| \equiv \langle Bx, x\rangle^{1/2}$.
- We have

$$L_1 \;\le\; 1, \qquad L_2 \;\le\; 2.$$

Log-sum-exp: convergence



Log-sum-exp: inner steps



Two ingredients

- Bregman divergence $\beta_d(v_k; y)$.
- Contraction operator

$$f(y) \quad \mapsto \quad f\left(\frac{a_{k+1}y + A_k x_k}{A_{k+1}}\right).$$

Direct acceleration vs. Proximal acceleration

- The rates are: $O\left(\frac{1}{k^{p+1}}\right)$ and $\tilde{O}\left(\frac{1}{k^{p+1}}\right)$, for the methods of order $p \geq 1$.
- In practice, the number of inner steps is a constant.
- Proximal acceleration is more general — useful for stochastic and distributed optimization.

Thank you for your attention!