Proximal Optimization with Automatic Dimension Reduction for Large Scale Learning

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Ph.D. Defence

3 November 2020

Supervised by F. IUTZELER, J. MALICK, and M.-R. AMINI

Motivation: Large Scale ML

Blackjack card counting

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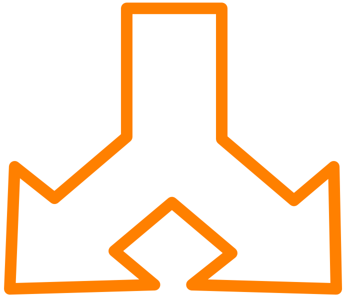
Are they counting?



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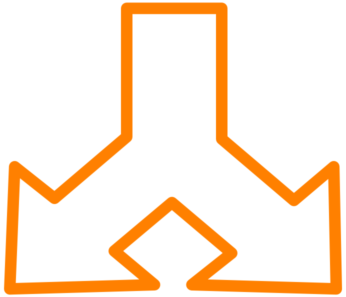

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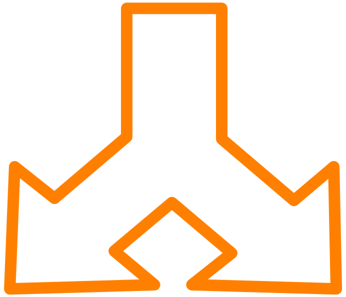

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Input: $(a_i, b_i)_{i=1,\dots,m} \in \mathcal{A} \times \{-1, 1\}$ - the set of observations.

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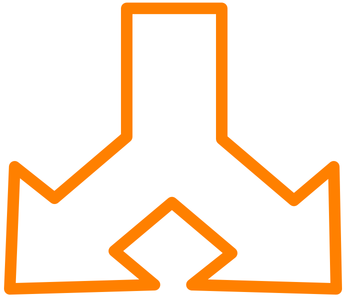

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Input: $(a_i, b_i)_{i=1,\dots,m} \in \mathcal{A} \times \{-1, 1\}$ - the set of observations.

Output: some prediction function $h(a, x)$ that belongs to some specific class.

ML as an Optimization Problem

Empirical Risk Minimization

ML as an Optimization Problem

Empirical Risk Minimization

Loss function: represents the difference between two arguments.

$$\underbrace{\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^{m} \ell(b_i, h(a_i, x))}_{f(x)}$$

ML as an Optimization Problem

Empirical Risk Minimization

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Learning is a compromise between accuracy and complexity

ML as an Optimization Problem

Structural Risk Minimization

Loss function: represents the difference between two arguments.

$$\min_{x \in \mathbb{R}^n} \underbrace{\frac{1}{m} \sum_{i=1}^{m} \ell(b_i, h(a_i, x))}_{f(x)} + r(x)$$

Regularization penalty.

ML as an Optimization Problem

UGA Université Grenoble Alpes

Structural Risk Minimization

Smooth, convex.

$$\min_{x\in\mathbb{R}^n} \underbrace{\frac{1}{m}\sum_{i=1}^{m} \ell(b_i, h(a_i, x))}_{f(x)} + r(x)$$

Convex, non-smooth.

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Sparse solution $r = \|\cdot\|_1,$

e.g. feature selection problems

Samuel Vaiter et al. *Model selection with low complexity priors.* Information and Inference: A Journal of the IMA 4.3 (2015): 230-287.

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Sparse solution $r = \|\cdot\|_1,$

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Fixed variation $r = \sum_{i=1}^{n-1} |x_{i+1} - x_i|.$

e.g. signal processing

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Proximal Gradient Descent

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Let us consider a composite optimization problem

$$\min_{x \in \mathbb{R}^n} f(x) + r(x),$$

where f is L-smooth and convex, and r is convex, l.s.c.

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$$\mathbf{prox}_r(y) = \operatorname*{argmin}_{x \in \mathbb{R}^n} \left\{ r(x) + \frac{1}{2}\|x - y\|_2^2 \right\}.$$

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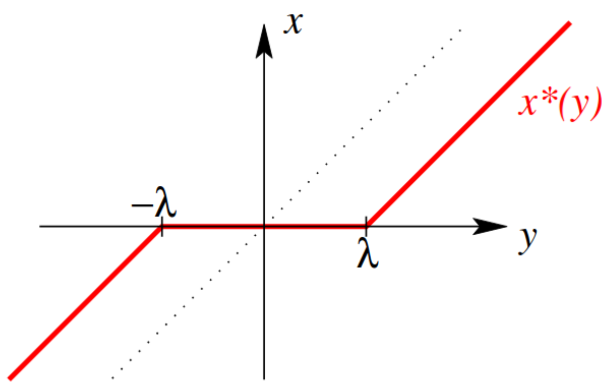
$r = \lambda\|\cdot\|_1$



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Step 2 $\quad x^{k+1} = \mathbf{prox}_{\gamma r}(y^k) \quad$ **backward (proximal) step.**

R Tyrrell Rockafellar. *Monotone operators and the proximal point algorithm.* SIAM journal on control and optimization, 14(5):877–898, 1976.

Identification

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One nice thing

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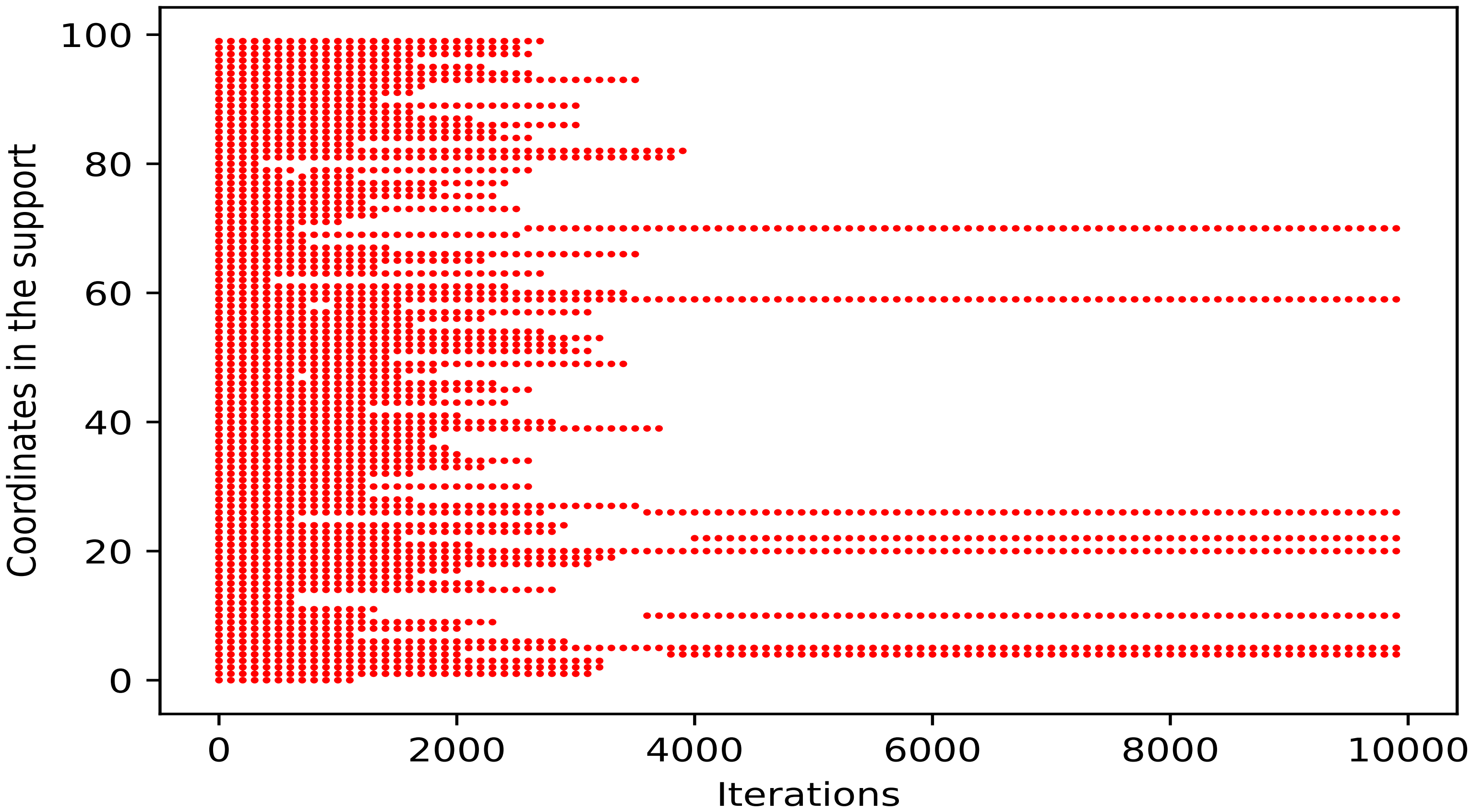
Proximal methods identify a near optimal subspace.



Synthetic LASSO problem $\min \frac{1}{2}\|Ax-b\|_2^2 + \lambda_1\|x\|_1$ for random generated matrix $A \in \mathbb{R}^{100\times 100}$ and vector $b \in \mathbb{R}^{100}$ and hyperparameter λ_1 chosen to reach 8% of density (amount of non-zero coordinates) of the final solution.

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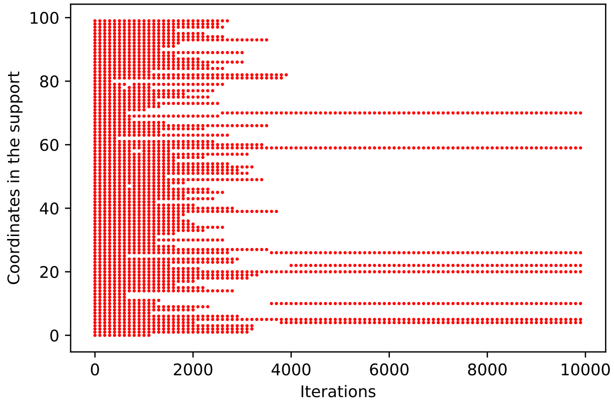

Sparsity vector

Let $\mathcal{M} = \{\mathcal{M}_1, \ldots, \mathcal{M}_m\}$ be a family of subspaces of $\mathbb{R}^n$ with m elements. We define the sparsity vector on $\mathcal{M}$ for point $x \in \mathbb{R}^n$ as the $\{0,1\}$-valued vector $\mathsf{S}_{\mathcal{M}}(x) \in \{0,1\}^m$ verifying

$$(\mathsf{S}_{\mathcal{M}}(x))_{[i]} = 0 \quad \text{if } x \in \mathcal{M}_i \text{ and } 1 \text{ elsewhere.}$$

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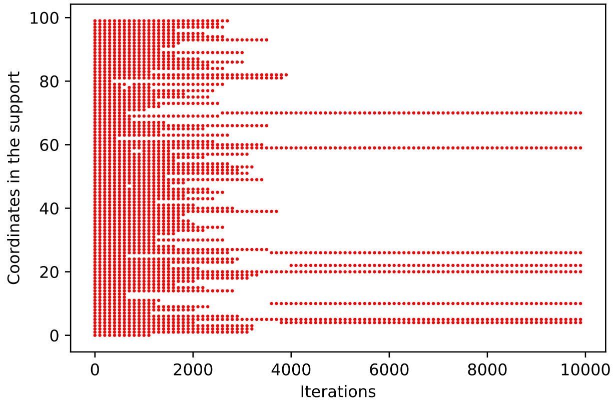

The collection $\mathcal{M} = \{\mathcal{M}_i\}_{1 \leq i \leq n}$ is the set of subspaces $\mathcal{M}_i$ with $\mathrm{supp}(x) = [n] \setminus \{i\}$ for all $x \in \mathcal{M}_i$.

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$$x^\star = \operatorname*{argmin}_{x\in\mathbb{R}^n} f(x) + r(x)$$

Université Grenoble Alpes

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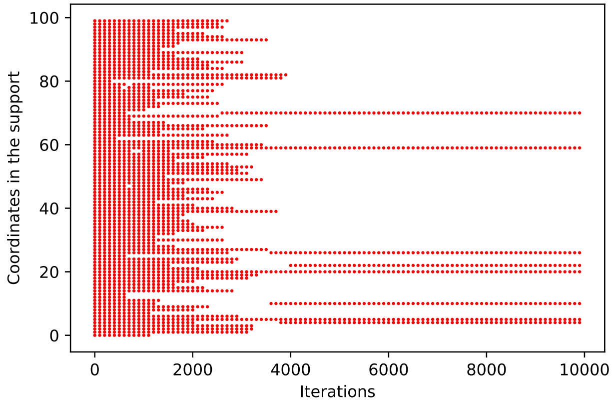

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Theorem (Enlarged identification)

Let (u^k) be an $\mathbb{R}^n$-valued sequence converging almost surely to $u^\star$ and define sequence (x^k) as $x^k = \mathbf{prox}_{\gamma r}(u^k)$ and $x^\star = \mathbf{prox}_{\gamma r}(u^\star)$. Then (x^k) identifies some subspaces with probability one; more precisely for any $\varepsilon > 0$, with probability one, after some finite time,

$$\mathsf{S}_{\mathcal{M}}(x^\star) \;\leq\; \mathsf{S}_{\mathcal{M}}(x^k) \;\leq\; \max\left\{\mathsf{S}_{\mathcal{M}}(\mathbf{prox}_{\gamma r}(u)) : u \in \mathcal{B}(u^\star, \varepsilon)\right\}.$$

Franck Iutzeler and Jérôme Malick. *Nonsmoothness in Machine Learning: specific structure, proximal identification, and applications.* Set-Valued and Variational Analysis (2020): 1-18.

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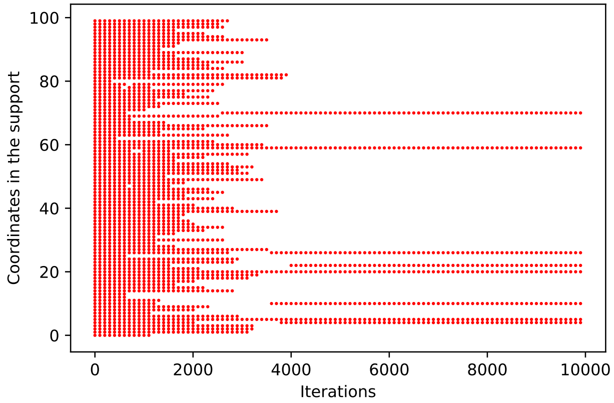

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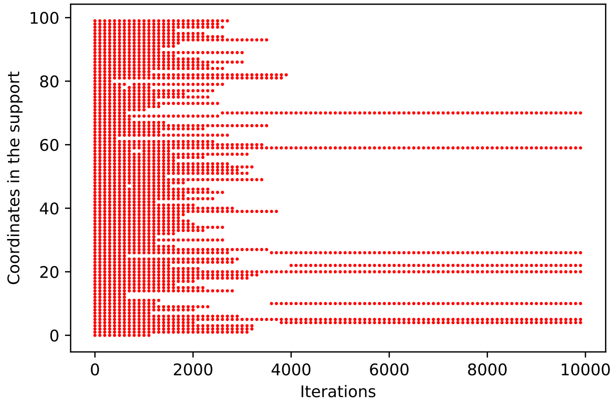

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The same = verifies QC.

Franck Iutzeler and Jérôme Malick. *Nonsmoothness in Machine Learning: specific structure, proximal identification, and applications.* Set-Valued and Variational Analysis (2020): 1-18.

Contributions

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– **Automatic dimension reduction**

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– **Identification based sparsification**

Contributions

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– **Reconditioned sparsification**

Contributions

– Automatic dimension reduction

Dmitry Grishchenko, Franck Iutzeler, and Jérôme Malick. *Proximal gradient methods with adaptive subspace sampling.* Mathematics of Operations Research, 2020.

In this part we consider a composite optimization problem

$$\min_{x\in\mathbb{R}^n} f(x) + r(x),$$

where f is L-smooth and μ-strongly convex, and r is convex, l.s.c. and prox-easy.

Randomized Coordinate Descent

Full gradient computation is expensive.

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Example 1 (smooth).

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Example 1 (smooth).

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Example 2 (separable regularizer).

$$r(x) = \sum_{i=1}^{n} r_i(x_{[i]}) \;\Rightarrow\; \mathbf{prox}_{\gamma r}(x)_{[i]} = \mathbf{prox}_{\gamma r_i}(x_{[i]}).$$

$$\mathrm{x}_{[i^k]}^{k+1} \leftarrow \mathbf{prox}_{\gamma r_{i^k}}\left(x_{[i^k]}^k - \gamma \nabla_{[i^k]} f(x^k)\right)$$

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Drawback: explicit use of the separability of the regularizer.

Peter Richtárik and Martin Takáč. *Iteration complexity of randomized block-coordinate descent methods for minimizing a composite function.* Mathematical Programming 144.1-2 (2014): 1-38.

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What if the regularizer is not separable?

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Olivier Fercoq and Pascal Bianchi. *A coordinate-descent primal-dual algorithm with large step size and possibly nonseparable functions.* SIAM Journal on Optimization 29.1 (2019): 100-134.

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In this reformulation the separability is not required!

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Two orthogonal projections
onto orthogonal spaces!

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$$x^{k+1} = \mathbf{prox}_{\gamma r}\left(P\left(y^k\right) + (I - P)\left(y^{k-1}\right)\right),$$

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Does it work like this?

Examples: Subspaces

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Let us consider the set of subspaces $\mathcal{C}_i$ such that $\mathcal{C}_i$ is i-th coordinate line. Select an orthogonal projection onto the $\mathcal{C}_i$ with probability $\frac{1}{n-1}$ $\forall i \in [2, n]$ and 0 for the 1-st.

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Does not work if the first coordinates of the starting and the optimal point are different.

Covering family of subspaces

Let $\mathcal{C} = \{\mathcal{C}_i\}_i$ be a family of subspaces of $\mathbb{R}^n$. We say that $\mathcal{C}$ is *covering* if it spans the whole space, i.e. if $\sum_i \mathcal{C}_i = \mathbb{R}^n$.

Admissible Selection

Admissible Selection

Let $\mathcal{C}$ be a covering family of subspaces of $\mathbb{R}^n$. A selection $\mathfrak{S}$ is defined from the set of all subsets of $\mathcal{C}$ to the set of the subspaces of $\mathbb{R}^n$ as

$$\mathfrak{S}(\omega) = \sum_{j=1}^{s} \mathcal{C}_{i_j} \qquad \text{for } \omega = \{\mathcal{C}_{i_1}, \ldots, \mathcal{C}_{i_s}\}.$$

The selection $\mathfrak{S}$ is *admissible* if $\mathbb{P}[x \in \mathfrak{S}^{\perp}] < 1$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

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If a selection $\mathfrak{S}$ is admissible then $\mathsf{P} := \mathbb{E}[P_{\mathfrak{S}}]$ is a positive definite matrix.

In this case, we denote by $\lambda_{\min}(\mathsf{P}) > 0$ and $\lambda_{\max}(\mathsf{P}) \leq 1$ its minimal and maximal eigenvalues.

Algorithm 1: RPSD

Algorithm 1 Randomized Proximal Subspace Descent - `RPSD`

Input: $\mathsf{Q} = \mathsf{P}^{-\frac{1}{2}}$

Initialize z^0, $x^1 = \mathbf{prox}_{\gamma r}(\mathsf{Q}^{-1}(z^0))$

for $k = 1, \ldots$ **do**

$\quad y^k = \mathsf{Q}\left(x^k - \gamma \nabla f\left(x^k\right)\right)$

$\quad z^k = P_{\mathfrak{S}^k}\left(y^k\right) + (I - P_{\mathfrak{S}^k})\left(z^{k-1}\right)$

$\quad x^{k+1} = \mathbf{prox}_{\gamma r}\left(\mathsf{Q}^{-1}\left(z^k\right)\right)$

end for

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2: Initialize z^0, $x^1 = \mathbf{prox}_{\gamma r}(\mathsf{Q}^{-1}(z^0))$
3: **for** $k = 1, \ldots$ **do**
4: $\quad y^k = \mathsf{Q}\left(x^k - \gamma \nabla f\left(x^k\right)\right)$
5: $\quad z^k = P_{\mathfrak{S}^k}\left(y^k\right) + (I - P_{\mathfrak{S}^k})\left(z^{k-1}\right)$ ← Project
6: $\quad x^{k+1} = \mathbf{prox}_{\gamma r}\left(\mathsf{Q}^{-1}\left(z^k\right)\right)$
7: **end for**

``Sketch''

RPSD: Convergence Result

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Assumption (on randomness)

Given a covering family $\mathcal{C} = \{\mathcal{C}_i\}$ of subspaces, we consider a sequence $\mathfrak{S}^1, \mathfrak{S}^2, .., \mathfrak{S}^k$ of admissible selections, which is i.i.d.

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Theorem (Convergence of RPSD)

For any $\gamma \in (0, 2/(\mu+L)]$, the sequence (x^k) of the iterates of `RPSD` converges almost surely to the minimizer $x^\star$ with rate

$$\mathbb{E}\left[\|x^{k+1} - x^\star\|_2^2\right] \leq \left(1 - \lambda_{\min}(\mathsf{P}) \frac{2\gamma\mu L}{\mu + L}\right)^k C,$$

where $C = \lambda_{\max}(\mathsf{P})\|z^0 - \mathsf{Q}(x^\star - \gamma \nabla f(x^\star))\|_2^2$.

RPSD: Convergence Result

Consider the set of subspaces $\mathcal{C}_i$ such that $\mathcal{C}_i$ is i-th coordinate line. Consider the selection $\mathfrak{S}$ such that $\mathbb{P}[\mathcal{C}_i \in \mathfrak{S}] = p_i > 0$, then $\lambda_{\min}(\mathsf{P}) = \min_i p_i > 0$.

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RPSD: Proof Sketch

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Lemma 1

From the minimizer $x^\star$, define the fixed points $z^\star = y^\star = \mathsf{Q}\left(x^\star - \gamma \nabla f\left(x^\star\right)\right)$ of the sequences (y^k) and (z^k). Then

$$\mathbb{E}\left[\|z^k - z^\star\|_2^2 \,|\, \mathcal{F}^{k-1}\right] = \|z^{k-1} - z^\star\|_2^2 + \|y^k - y^\star\|_{\mathsf{P}}^2 - \|z^{k-1} - z^\star\|_{\mathsf{P}}^2,$$

where $\mathcal{F}^k = \sigma(\{\mathfrak{S}_\ell\}_{\ell \leq k})$ is the filtration of the past random subspaces.

$$z^k = P_{\mathfrak{S}^k}\left(y^k\right) + \left(I - P_{\mathfrak{S}^k}\right)\left(z^{k-1}\right)$$

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Lemma 2

Using the same notations as in Lemma 1

$$\|y^k - y^\star\|_{\mathsf{P}}^2 - \|z^{k-1} - z^\star\|_{\mathsf{P}}^2 \leq -\lambda_{\min}(\mathsf{P})\frac{2\gamma\mu L}{\mu + L}\|z^{k-1} - z^\star\|_2^2.$$

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Identification!

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Examples: TV Projections

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$$r = \lambda \sum_{i=1}^{n-1} \left| x_{[i]} - x_{[i+1]} \right|$$

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Projection on such set

$$P_{\mathfrak{S}} = \begin{array}{cc}
\overbrace{\hphantom{\tfrac{1}{n_1}\ \cdots\ \tfrac{1}{n_1}}}^{n_1}\hphantom{\ 0\ \cdots\ }\overbrace{\hphantom{\cdots\ \cdots\ 0}}^{n-n_s} & \\
\begin{pmatrix}
\frac{1}{n_1} & \cdots & \frac{1}{n_1} & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\frac{1}{n_1} & \cdots & \frac{1}{n_1} & 0 & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 & \frac{1}{n-n_s} & \cdots & \frac{1}{n-n_s} \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
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\end{pmatrix} &
\begin{array}{l}
\left.\vphantom{\begin{matrix}\frac{1}{n_1}\\ \vdots \\ \frac{1}{n_1}\end{matrix}}\right\} n_1 \\
\vphantom{\begin{matrix}0\\ \vdots\end{matrix}} \\
\left.\vphantom{\begin{matrix}\frac{1}{n-n_s}\\ \vdots \\ \frac{1}{n-n_s}\end{matrix}}\right\} n-n_s
\end{array}
\end{array}$$

Algorithm 2: ARPSD

Algorithm 2 Adaptive Randomized Proximal Subspace Descent - `ARPSD`

Initialize z^0, $x^1 = \mathbf{prox}_{\gamma g}(\mathsf{Q}_0^{-1}(z^0))$, $\ell = 0$, $\mathsf{L} = \{0\}$.
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    $\mathsf{L} \leftarrow \mathsf{L} \cup \{k+1\}$, $\ell \leftarrow \ell + 1$
    Generate a new admissible selection
    Compute $\mathsf{Q}_\ell = \mathsf{P}_\ell^{-\frac{1}{2}}$ and Q_ℓ^{-1}
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Université Grenoble Alpes

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$$\begin{cases} (\mathcal{C}_i \bigcap \mathcal{M}_i) \subseteq \bigcap_j \mathcal{C}_j \\ \mathcal{C}_i + \mathcal{M}_i = \mathbb{R}^n \end{cases}$$

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$$\mathbb{P}[\mathcal{C}_i \in \mathfrak{S}^{k+1}] = \begin{cases} p & \text{if } x^{k+1} \in \mathcal{M}_i \Leftrightarrow [\mathsf{S}_\mathcal{M}(x^{k+1})]_i = 0 \\ 1 & \text{elsewhere} \end{cases}$$

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Adaptation Process

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Let us specify ARPSD with the following simple adaptation strategy. We take a fixed upper bound on the adaptation cost and a fixed lower bound on uniformity:

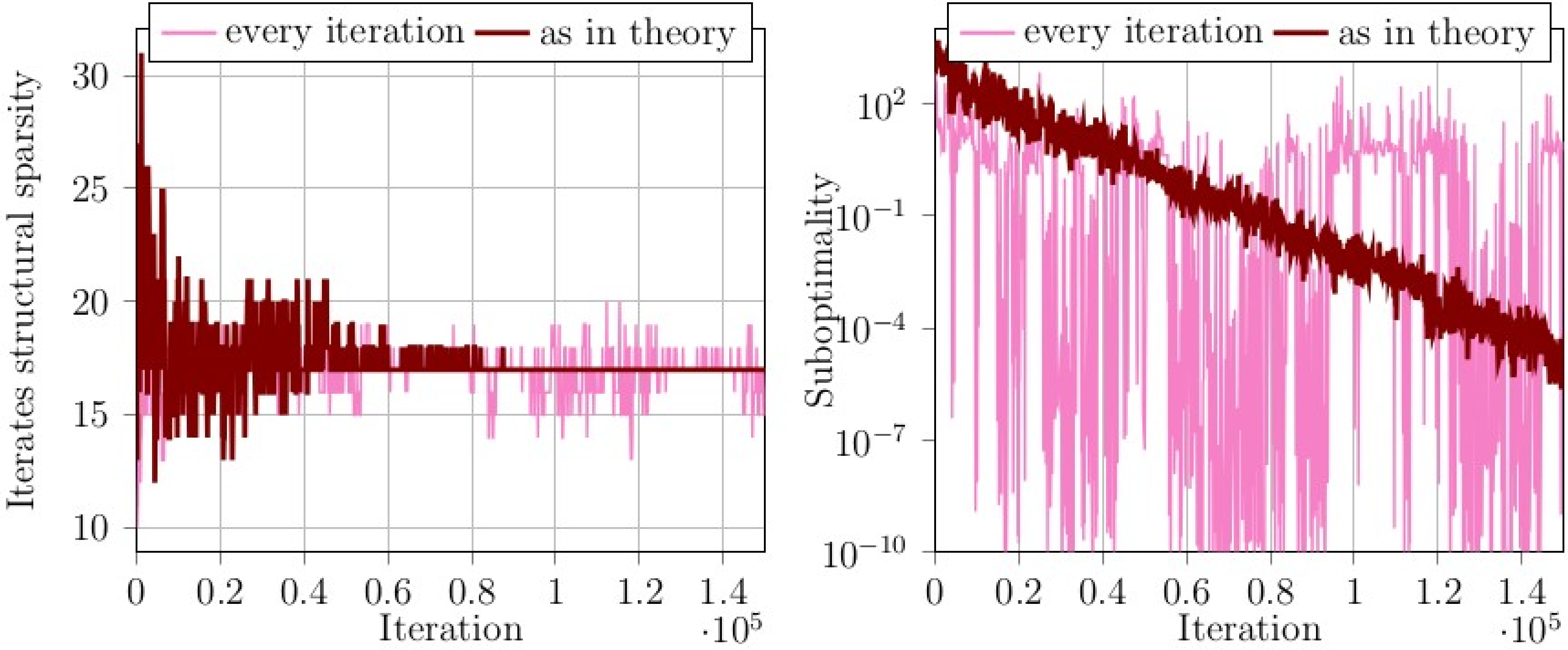
$$\|\mathsf{Q}_\ell \mathsf{Q}_{\ell-1}^{-1}\|_2^2 \leq \mathbf{a} \qquad \lambda_{\min}(\mathsf{P}_\ell) \geq \lambda.$$

Then from the rate $1-\alpha = 1 - 2\gamma\mu L\lambda/(\mu+L)$, we can perform an adaptation every

$$\mathbf{c} = \left\lceil \log(\mathbf{a}) / \log\big((2-\alpha)/(2-2\alpha)\big) \right\rceil$$

iterations, so that $\mathbf{a}(1-\alpha)^{\mathbf{c}} = (1-\alpha/2)^{\mathbf{c}}$ and $k_\ell = \ell\mathbf{c}$.

Adaptation Process



ARPSD: Convergence Result

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Assumption (on randomness)

For all $k > 0$, $\mathfrak{S}^k$ is $\mathcal{F}^k$-measurable and admissible. Furthermore, if $k \notin \mathsf{L}$, $(\mathfrak{S}^k)$ is independent and identically distributed on $[k_\ell, k]$. The decision to adapt or not at time k is $\mathcal{F}^k$-measurable, i.e. $(k_\ell)_\ell$ is a sequence of $\mathcal{F}^k$-stopping times.

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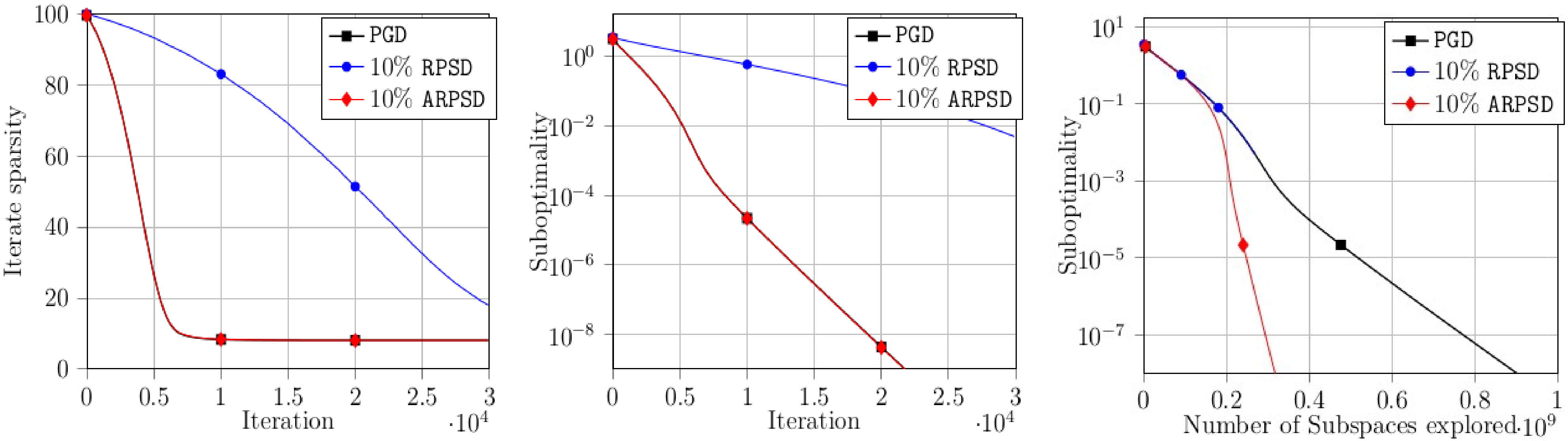
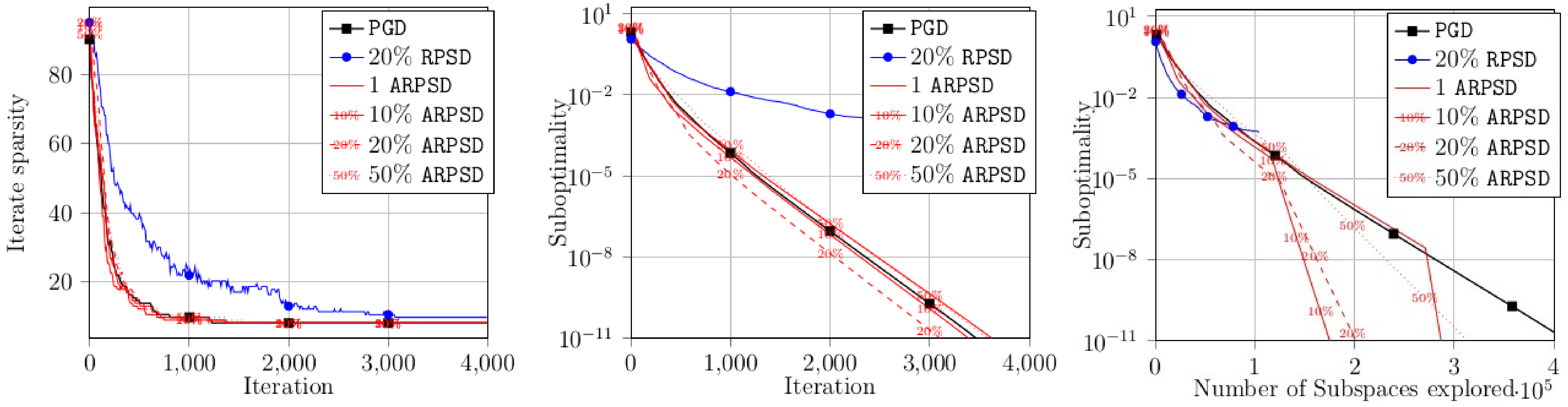
Experiments: Inefficiency of RPSD



Logistic regression with elastic net regularizer on rcv1_train dataset ($n = 47236$ $m = 20242$).

$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{m} \sum_{j=1}^{m} \log(1+\exp(-y_j z_j^\top x)) + \lambda_1 \|x\|_1 + \frac{\lambda_2}{2} \|x\|_2^2$$

Experiments: ARPSD with TV



1D-TV-regularized logistic regression on a1a dataset ($n = 123$ $m = 1605$).

$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{m}\sum_{j=1}^{m} \log(1+\exp(-y_j z_j^\top x)) + \lambda_1 \sum_{i=1}^{n-1} |x_{[i]} - x_{[i+1]}| + \frac{\lambda_2}{2}\|x\|_2^2$$

Strange Metric?

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$$\min_{x \in \mathbb{R}^n} \underbrace{\frac{1}{m} \sum_{i=1}^{m} \ell(b_i, h(a_i, x))}_{f(x)} + r(x)$$

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$$\min_{x\in\mathbb{R}^n} \sum_{i=1}^{M} \alpha_i \underbrace{\left[\frac{1}{|\mathcal{D}_i|}\sum_{j\in\mathcal{D}_i} \ell(b_j, h(a_j, x))\right]}_{f_i} + r(x),$$

where the full dataset $\mathcal{D}$ is split onto M nonintersecting subsets $\mathcal{D}_i$ and α_i is the proportion of examples $\frac{|\mathcal{D}_i|}{m}$.

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$$z^k = \sum_i \alpha_i z_i^k \xrightarrow{\text{Communication}}$$

Master

$$z_i^k = P_{\mathfrak{S}^k}\left(y_i^k\right) + \left(I - P_{\mathfrak{S}^k}\right)\left(z_i^{k-1}\right)$$

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Bottleneck

Amount of subspaces explored

Master

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Contributions

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– Identification based sparsification

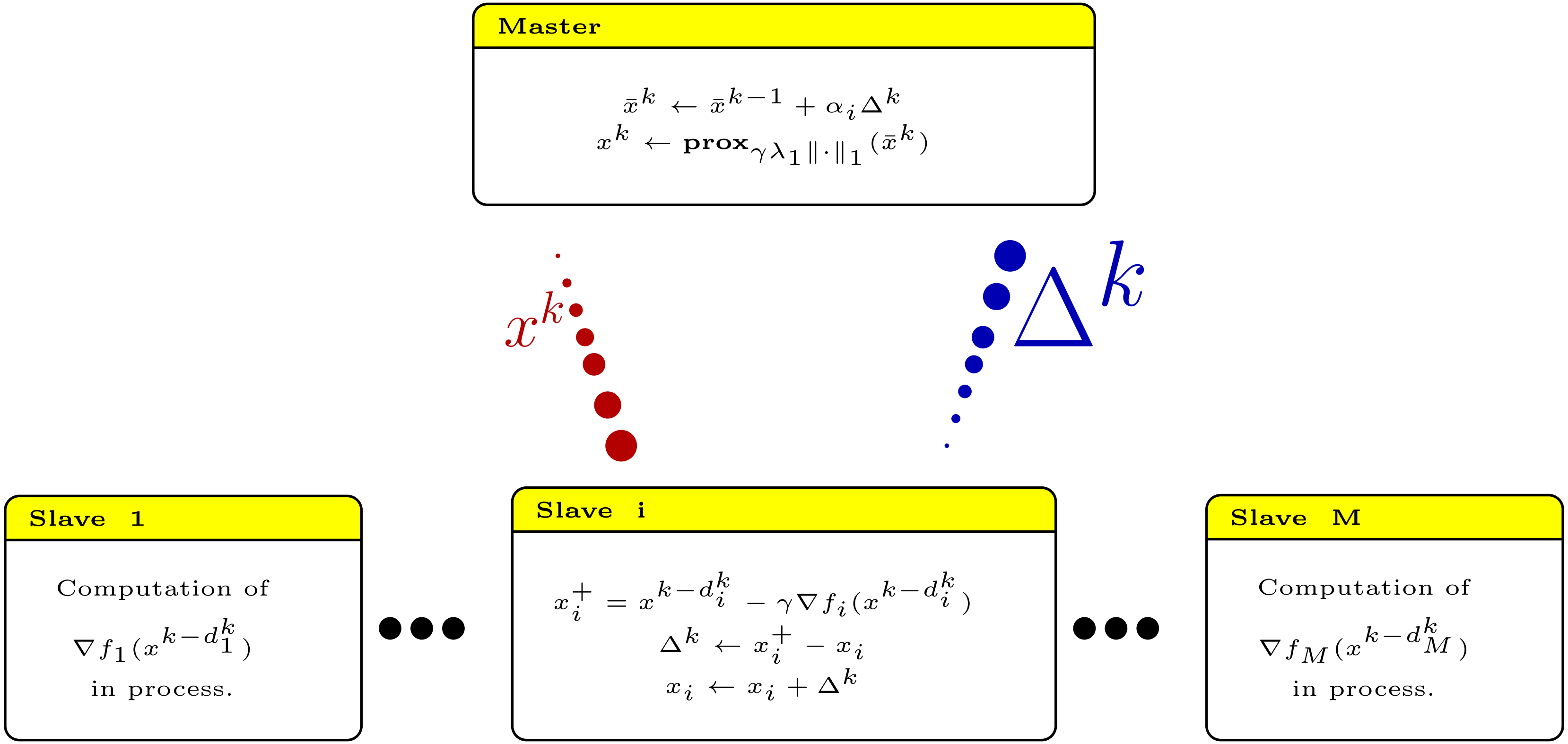
Dmitry Grishchenko, Franck Iutzeler, and Massih-Reza Amini. *Sparse Asynchronous Distributed Learning*, International Conference on Neural Information Processing 2020.

In the next two parts we consider asynchronous distributed setup where m observations are split down over M machines, each machine i having a private subset $\mathcal{D}_i$ of the examples

$$\min_{x\in\mathbb{R}^n} \; F(x) = \sum_{i=1}^{M} \alpha_i f_i(x) + \lambda_1 \|x\|_1,$$

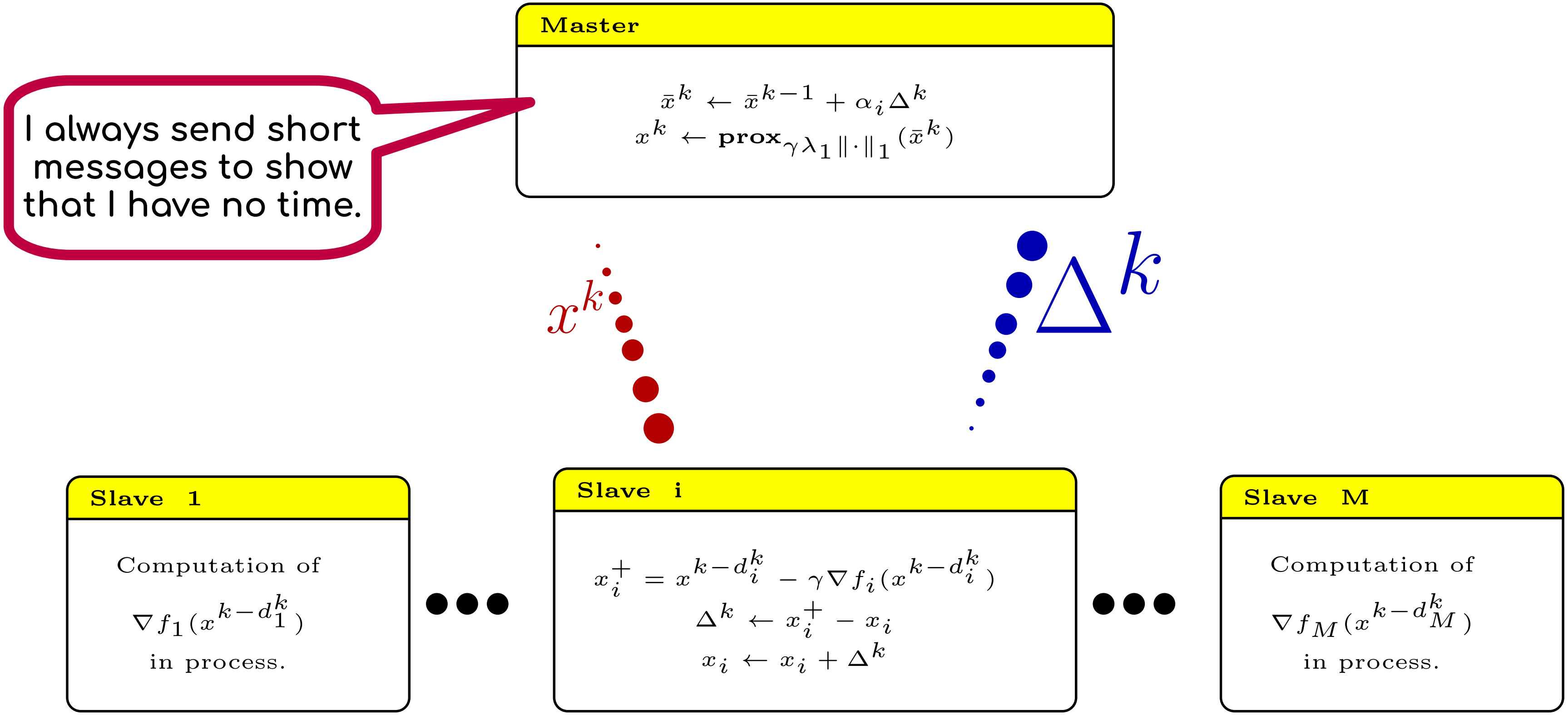
with $\alpha_i = |\mathcal{D}_i|/m$ being the proportion of observations locally stored in machine i, hence functions (f_i) are L-smooth and μ-strongly convex.

Algorithm: DAve-PG



Konstantin Mishchenko, Franck Iutzeler, Jérôme Malick, and Massih-Reza Amini. *A Delay-tolerant Proximal-Gradient Algorithm for Distributed Learning*, International Conference on Machine Learning, 3584-3592

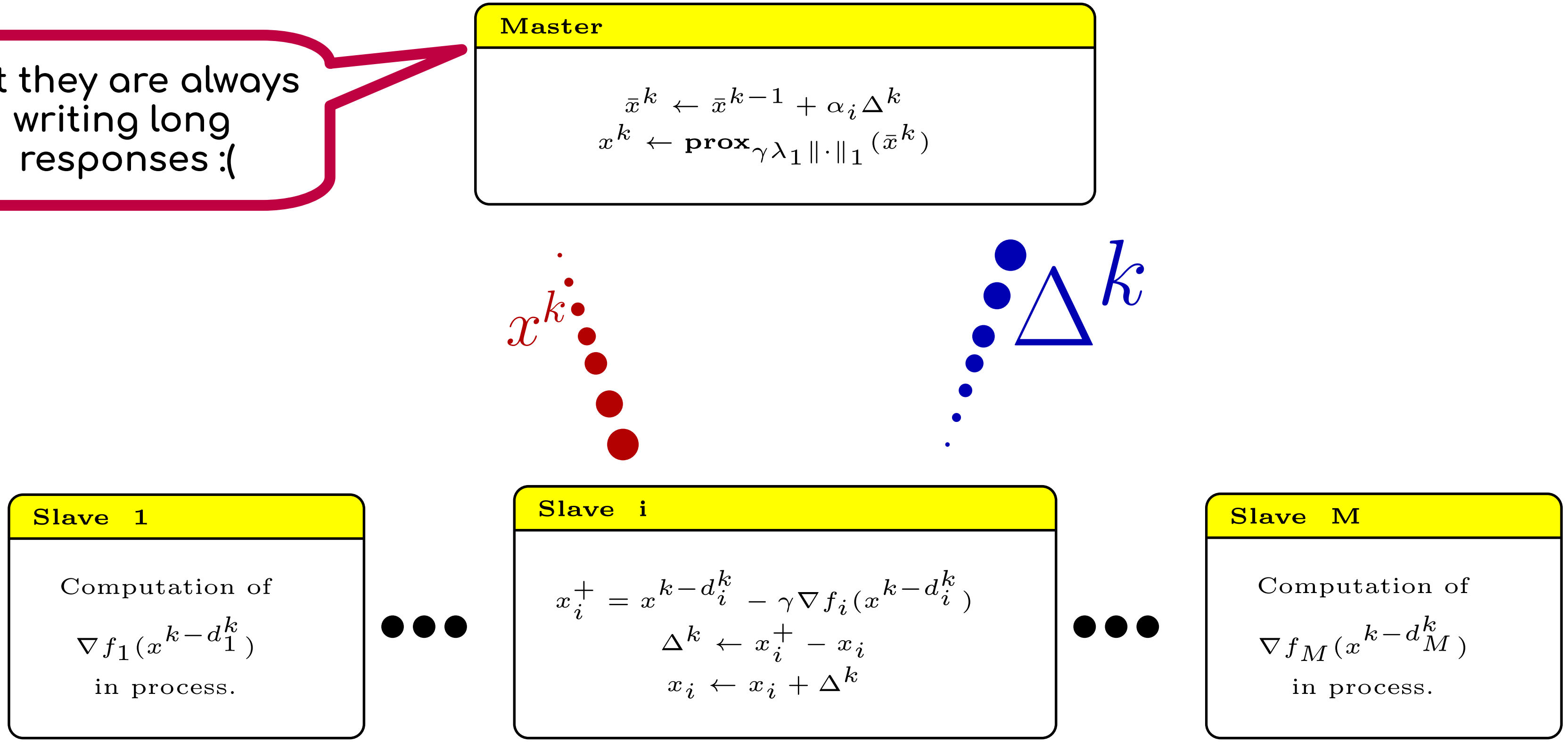
Algorithm: DAve-PG



Konstantin Mishchenko, Franck Iutzeler, Jérôme Malick, and Massih-Reza Amini. *A Delay-tolerant Proximal-Gradient Algorithm for Distributed Learning*, International Conference on Machine Learning, 3584-3592

Algorithm: DAve-PG

But they are always writing long responses :(



Konstantin Mishchenko, Franck Iutzeler, Jérôme Malick, and Massih-Reza Amini. *A Delay-tolerant Proximal-Gradient Algorithm for Distributed Learning*, International Conference on Machine Learning, 3584-3592

Algorithm: DAve-PG

Université Grenoble Alpes

Master

$$\bar{x}^k \leftarrow \bar{x}^{k-1} + \alpha_i \Delta^k$$
$$x^k \leftarrow \mathbf{prox}_{\gamma\lambda_1\|\cdot\|_1}(\bar{x}^k)$$

I am going to ignore messages that are twice longer than mine!

x^k Δ^k

Slave 1

Computation of

$\nabla f_1(x^{k-d_1^k})$

in process.

•••

Slave i

$$x_i^+ = x^{k-d_i^k} - \gamma \nabla f_i(x^{k-d_i^k})$$
$$\Delta^k \leftarrow x_i^+ - x_i$$
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•••

Slave M

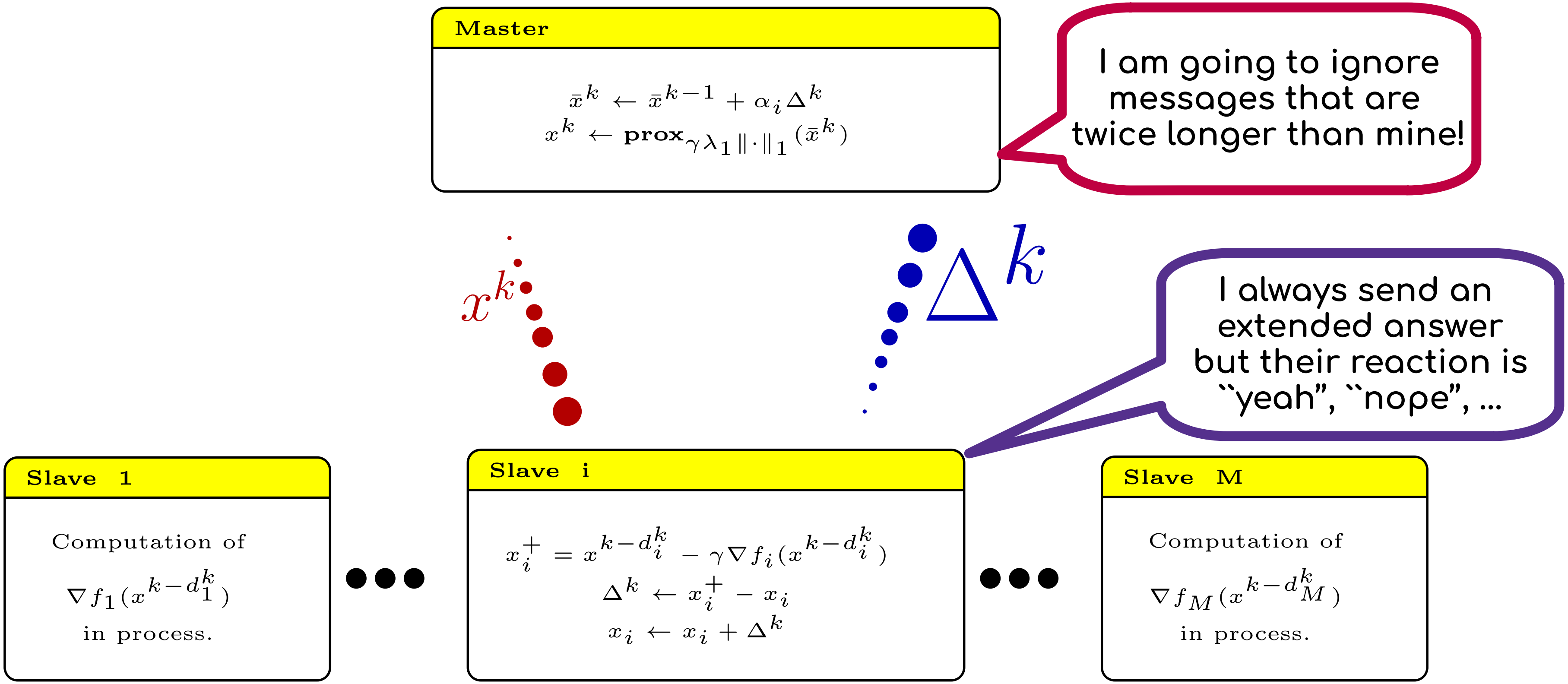
Computation of

$\nabla f_M(x^{k-d_M^k})$

in process.

Konstantin Mishchenko, Franck Iutzeler, Jérôme Malick, and Massih-Reza Amini. *A Delay-tolerant Proximal-Gradient Algorithm for Distributed Learning*, International Conference on Machine Learning, 3584-3592

Algorithm: DAve-PG



Konstantin Mishchenko, Franck Iutzeler, Jérôme Malick, and Massih-Reza Amini. *A Delay-tolerant Proximal-Gradient Algorithm for Distributed Learning*, International Conference on Machine Learning, 3584-3592

Algorithm: SPY

Master

$$\bar{x}^k \leftarrow \bar{x}^{k-1} + \alpha_i \Delta^k$$
$$x^k \leftarrow \mathbf{prox}_{\gamma\lambda_1\|\cdot\|_1}(\bar{x}^k)$$

I am going to ignore messages that are twice longer than mine!

From now on I am going to send as short messages as theirs.

x^k

$[\Delta^k]_{\mathbf{S}^k}$

Slave 1

Computation of

$\nabla f_1(x^{k-d_1^k})$

in process.

•••

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$$x_i^+ = x^{k-d_i^k} - \gamma \nabla f_i(x^{k-d_i^k})$$
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Slave M

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in process.

Examples: Mask Selection

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$$\left[\Delta^k\right]_{\mathbf{S}^k}$$

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Random sparsificaiton with $p = (p_1, .., p_n) \in (0, 1]^n$.

$$\mathbb{P}[j \in \mathbf{S}_p^k] = p_j > 0 \qquad \text{for all } j \in \{1, .., n\}.$$

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Examples: Mask Selection

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Random sparsificaiton with $p = (p_1, .., p_n) \in (0, 1]^n$.

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- p is an arbitrary probability vector.
- p is a π-uniform probability vector.
- p is a π-priority random vector w.r.t. some point x

$$\mathbb{P}\left[j \in \mathbf{S}_\pi^k\right] = \begin{cases} 1 & \text{if j} \in \text{supp}(x), \\ \pi & \text{otherwise.} \end{cases}$$

General Theoretical Result

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Assumption (on randomness)

The sparsity mask selectors $(\mathbf{S}_p^k)$ are independent and identically distributed random variables. We select a coordinate in the mask as follows:

$$\mathbb{P}[j \in \mathbf{S}_p^k] = p_j > 0 \qquad \text{for all } j \in \{1, .., n\},$$

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Theorem (Limits of sparsification)

Take $\gamma = \frac{2}{\mu+L}$, then $\texttt{SPY}$ verifies for all $k \in [k_m, k_{m+1})$

$$\mathbb{E}\left\|x^k - x^\star\right\|^2 \leq \left(p_{\max}\left(\frac{1-\kappa_{\mathrm{P}}}{1+\kappa_{\mathrm{P}}}\right)^2 + 1 - p_{\min}\right)^m \max_i \left\|x_i^0 - x_i^\star\right\|^2.$$

with the shifted local solutions $x_i^\star = x^\star - \gamma_i \nabla f_i(x^\star)$.

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Theorem (Limits of sparsification)

Take $\gamma = \frac{2}{\mu+L}$, then `SPY` with π-uniform sampling verifies for all $k \in [k_m, k_{m+1})$

$$\mathbb{E}\left\|x^k - x^\star\right\|^2 \leq \left(1 - \pi\frac{4\mu L}{(\mu+L)^2}\right)^m \max_i \left\|x_i^0 - x_i^\star\right\|^2.$$

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The sparsity mask selectors $(\mathbf{S}_p^k)$ are independent and identically distributed random variables. We select a coordinate in the mask as follows:

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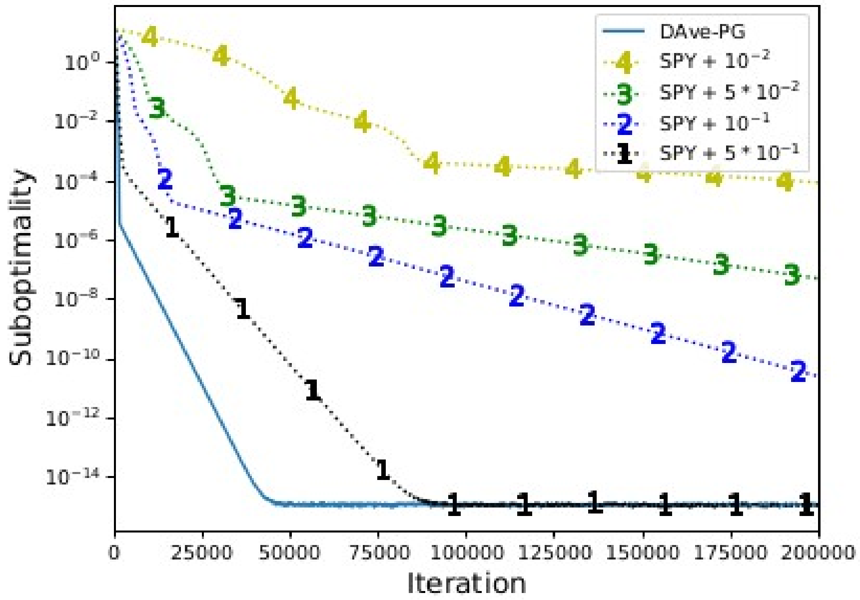
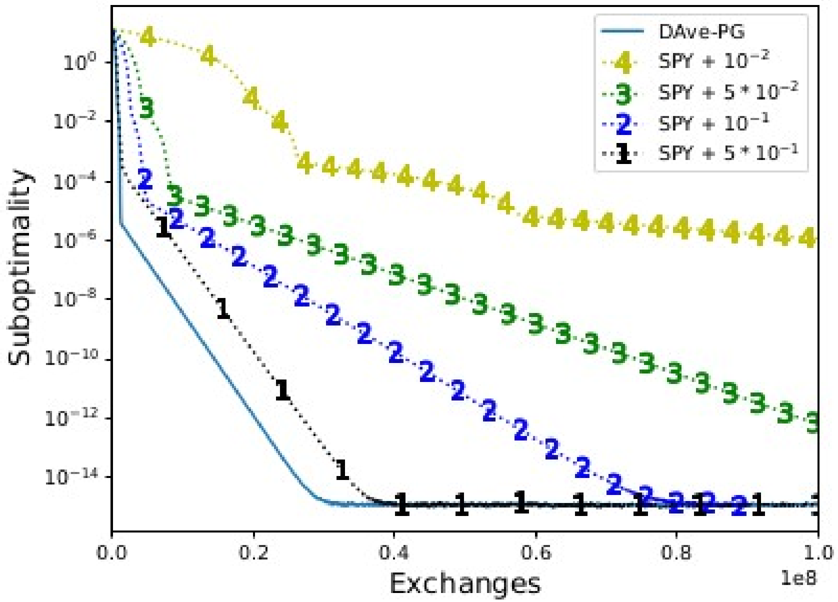
Limits of sparsification

`SPY` reaches linear convergence of the mean squared error in terms of epochs if

$$\frac{p_{\min}}{p_{\max}} > (1 - \gamma\mu)^2 \overset{\gamma = \frac{2}{\mu + L}}{\geq} \left(\frac{1 - \kappa_{\mathrm{P}}}{1 + \kappa_{\mathrm{P}}}\right)^2 .$$

Experiments: Uniform Sampling





Logistic regression with elastic net regularizer on madelon dataset ($n = 500$ $m = 2000$) and $M = 10$ machines.

$$\min_{x\in\mathbb{R}^n} \quad \frac{1}{m}\sum_{j=1}^{m} \log(1+\exp(-y_j z_j^\top x)) + \lambda_1\|x\|_1 + \frac{\lambda_2}{2}\|x\|_2^2$$

Experiments: Adaptive Selection

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p is π-priority random vector w.r.t. the current iterate point x^k

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This selection is not i.i.d.!

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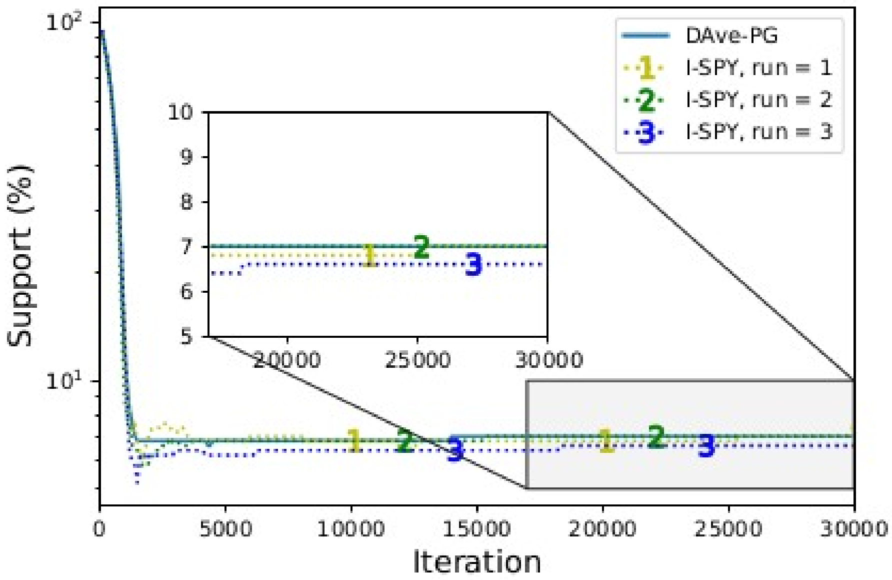
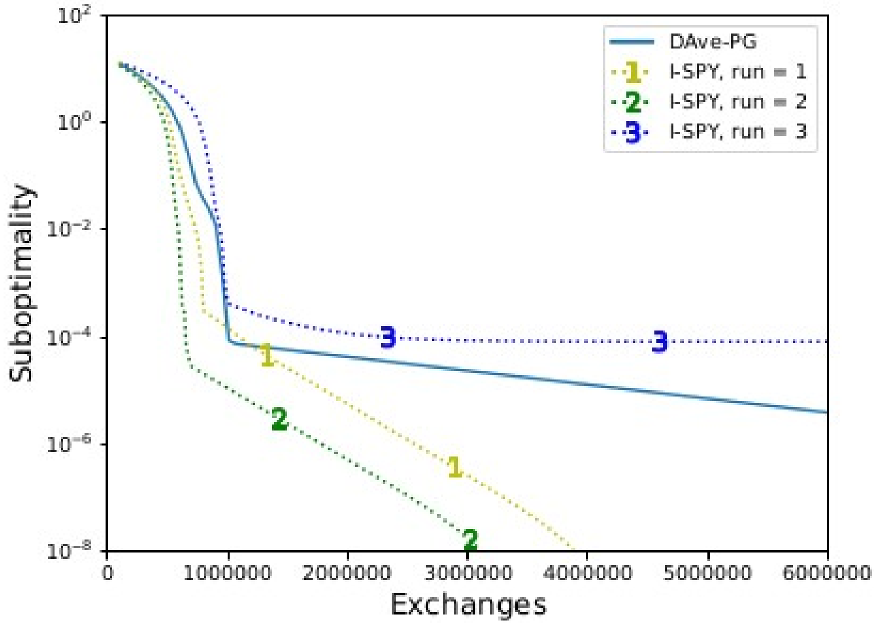
This selection is not i.i.d.!

If support is fixed the selection is i.i.d.!

Experiments: Adaptive Selection





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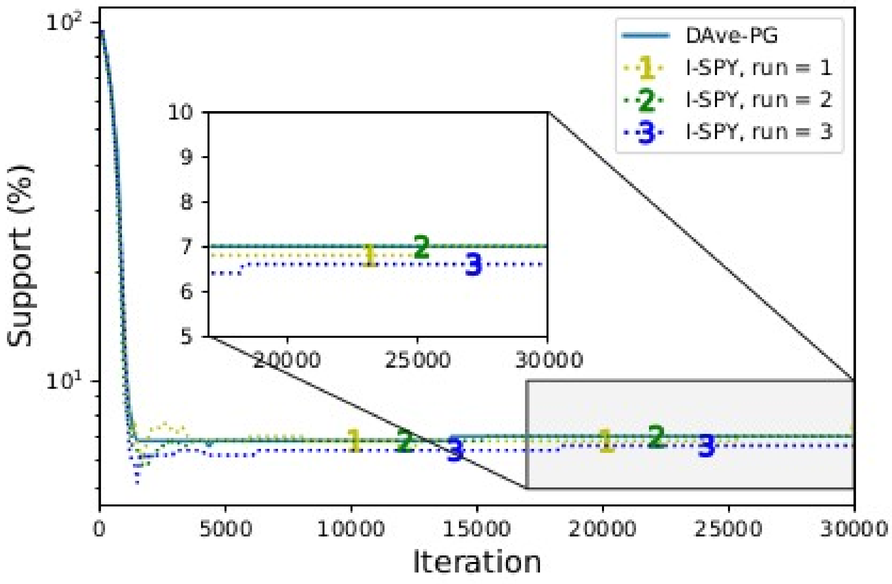

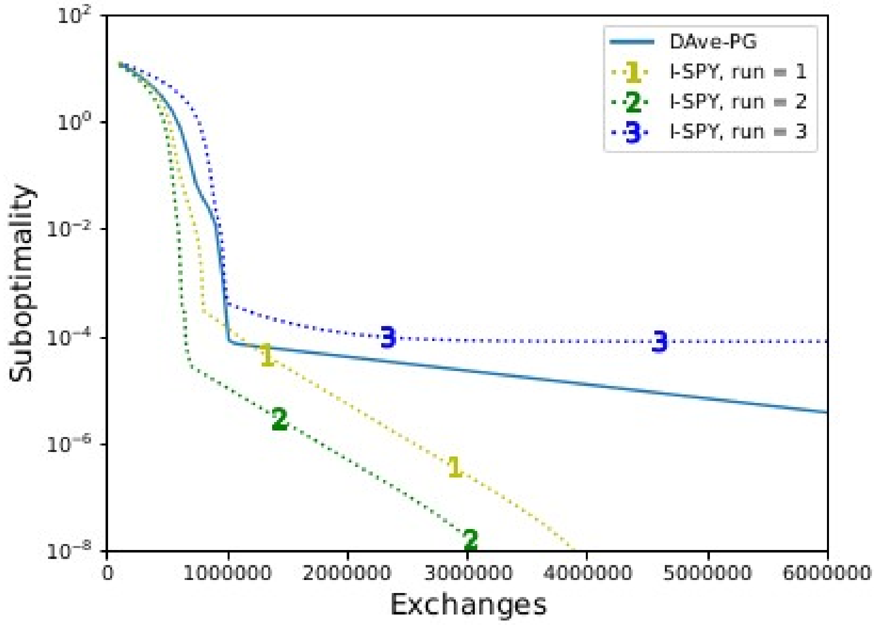

It is better if it converges, but it can diverge!

Contributions

– Reconditioned sparsification

Dmitry Grishchenko, Franck Iutzeler, Jérôme Malick, and Massih-Reza Amini. *Distributed Learning with Automatic Compression by Identification,* Submitted to SIMODS.

Proximal Reconditioning

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Adaptive mask selection can be used safely only for well-conditioned problems.

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Ill-conditioned problem ⇒ **Well-conditioned problem**

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Proximal reconditioning

Ill-conditioned problem $\Longrightarrow$ **Well-conditioned problem**

Accelerated proximal reconditioning

A. Ivanova D. Pasechnyuk, D. Grishchenko, E. Shulgin, A. Gasnikov, V. Matyukhin. *Adaptive catalyst for smooth convex optimization.* Submitted to OMS.

Lin, Hongzhou, Julien Mairal, and Zaid Harchaoui *A universal catalyst for first-order optimization.* Advances in neural information processing systems. 2015.

Proximal Reconditioning

Adaptive mask selection can be used safely only for well-conditioned problems.

Ill-conditioned problem $\xrightarrow{\text{Proximal reconditioning}}$ **Well-conditioned problem**

Proximal Reconditioning

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i-th worker function: f_i

Proximal Reconditioning

~~i-th worker function: f_i~~

i-th worker NEW function: $h_{i,\ell} = f_i + \frac{\rho}{2}\| \cdot - x_\ell\|_2^2$, where ℓ corresponds to the outer loop.

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$$\kappa = \frac{\mu + \rho}{L + \rho} \quad \left(\geq \ \kappa_{\mathsf{P}} = \frac{\mu}{L} \right).$$

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New problem

$$\mathbf{prox}_{F/\rho}(x_\ell) = \operatorname*{argmin}_{x\in\mathbb{R}^n} \underbrace{\sum_{i=1}^{M} \alpha_i f_i(x) + \lambda_1 \|x\|_1}_{=F(x)} + \frac{\rho}{2}\|x - x_\ell\|_2^2.$$

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Outer loop

Reconditioned Algorithm

Initialize x_1, $n \geq c > 0$, and $\delta \in (0,1)$.

Set $\rho = \dfrac{\kappa L - \mu}{1-\kappa}$ and $\gamma \in \left(0, \dfrac{2}{\mu + L + 2\rho}\right]$ with $\kappa = \dfrac{1-\sqrt{\pi-\alpha}}{1+\sqrt{\pi-\alpha}}$; $\pi = \dfrac{c}{n}$ and $\alpha = \dfrac{c}{2n}$.

while *the desired accuracy is not achieved* **do**

Observe the support of x_ℓ, compute p_ℓ as

$$p_{j,\ell} = \begin{cases} \pi_\ell := \min\left(\frac{c}{|\mathrm{null}(x_\ell)|}; 1\right) & \text{if } [x_\ell]_j = 0 \\ 1 & \text{if } [x_\ell]_j \neq 0 \end{cases} \quad \text{for all } j \in \{1, \ldots, n\}.$$

Compute an approximate solution of the reconditioned problem with `I-SPY`

$$x_{\ell+1} \approx \mathbf{prox}_{F/\rho}(x_\ell) = \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ \sum_{i=1}^{M} \alpha_i \underbrace{\left(f_i(x) + \frac{\rho}{2}\|x - x_\ell\|_2^2\right)}_{h_{i,\ell}(x)} + r(x) \right\}$$

with p_ℓ and x_ℓ as initial point. Stopping criterion is fixed budget

$$\mathsf{M}_\ell = \left[\frac{(1+\delta)\log(\ell)}{\log\left(\frac{1}{1-\alpha+\pi-\pi_\ell}\right)} + \frac{\log\left(\frac{2\mu+\rho}{(1-\delta)\rho}\right)}{\log\left(\frac{1}{1-\alpha+\pi-\pi_\ell}\right)} \right] \text{ epochs.}$$

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Compute an approximate solution of the reconditioned problem with `I-SPY`

linearly converges

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Compute an approximate solution of the reconditioned problem with `I-SPY`

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identification (inner)

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Compute an approximate solution of the reconditioned problem with `I-SPY`

linearly converges

identification (inner)

linearly converges to the optimal point

$$x_{\ell+1} \approx \mathbf{prox}_{F/\rho}(x_\ell) = \operatorname*{argmin}_{x \in \mathbb{R}^n} \left\{ \sum_{i=1}^{M} \alpha_i \underbrace{\left(f_i(x) + \frac{\rho}{2}\|x - x_\ell\|_2^2\right)}_{h_{i,\ell}(x)} + r(x) \right\}$$

with p_ℓ and x_ℓ as initial point. Stopping criterion is fixed budget

$$\mathsf{M}_\ell = \left\lceil \frac{(1+\delta)\log(\ell)}{\log\left(\frac{1}{1-\alpha+\pi-\pi_\ell}\right)} + \frac{\log\left(\frac{2\mu+\rho}{(1-\delta)\rho}\right)}{\log\left(\frac{1}{1-\alpha+\pi-\pi_\ell}\right)} \right\rceil \text{ epochs.}$$

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Observe the support of x_ℓ, compute p_ℓ as

$$p_{j,\ell} = \begin{cases} \pi_\ell := \min\left(\frac{c}{|\mathrm{null}(x_\ell)|}; 1\right) & \text{if } [x_\ell]_j = 0 \\ 1 & \text{if } [x_\ell]_j \neq 0 \end{cases} \quad \text{for all } j \in \{1, \ldots, n\}.$$

Compute an approximate solution of the reconditioned problem with `I-SPY`

linearly converges

identification (inner)

linearly converges to the optimal point

$$x_{\ell+1} \approx \mathbf{prox}_{F/\rho}(x_\ell) = \operatorname*{argmin}_{x\in\mathbb{R}^n} \left\{ \sum_{i=1}^{M} \alpha_i \underbrace{\left(f_i(x) + \frac{\rho}{2}\|x - x_\ell\|_2^2\right)}_{h_{i,\ell}(x)} + r(x) \right\}$$

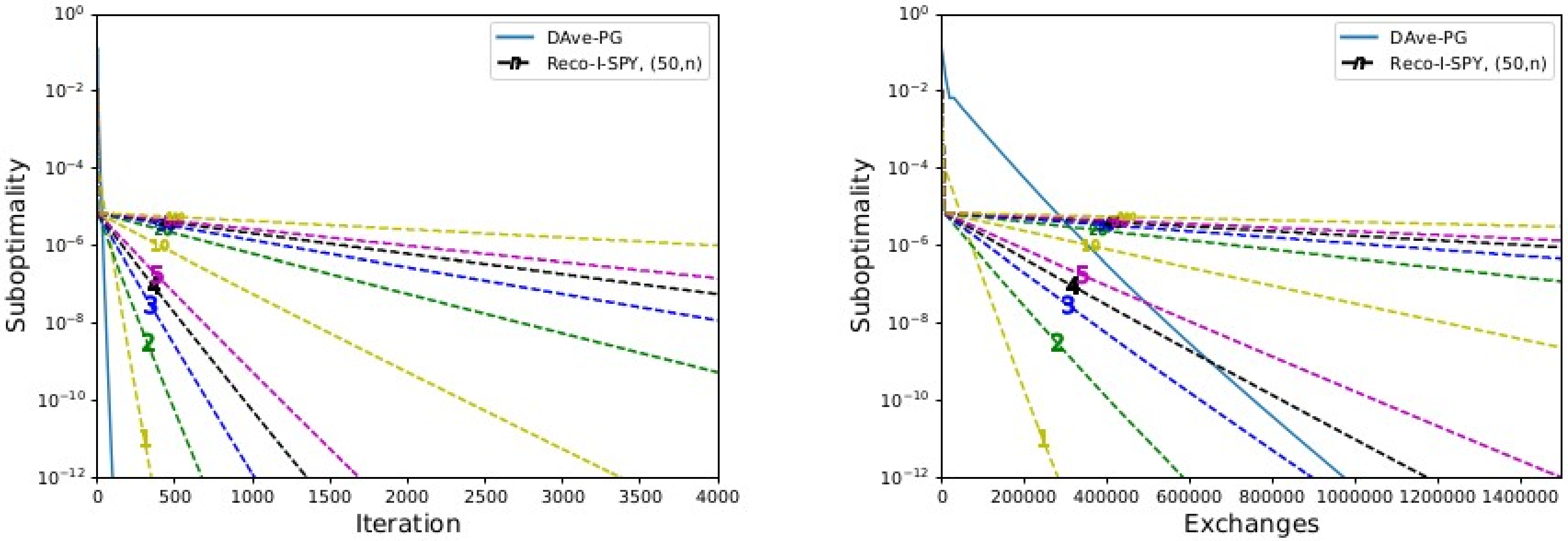
identification (global)

with p_ℓ and x_ℓ as initial point. Stopping criterion is fixed budget

$$\mathsf{M}_\ell = \left\lceil \frac{(1+\delta)\log(\ell)}{\log\left(\frac{1}{1-\alpha+\pi-\pi_\ell}\right)} + \frac{\log\left(\frac{2\mu+\rho}{(1-\delta)\rho}\right)}{\log\left(\frac{1}{1-\alpha+\pi-\pi_\ell}\right)} \right\rceil \text{ epochs.}$$

end

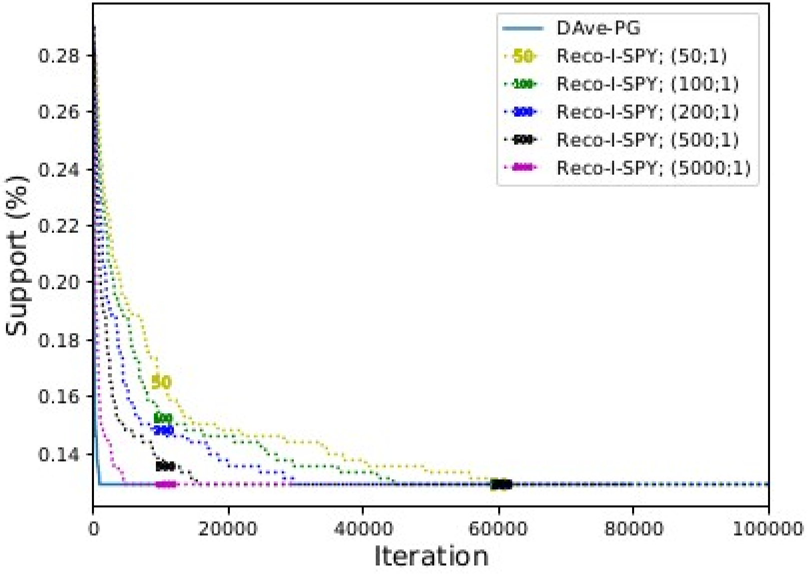
Experiments: Different Budget

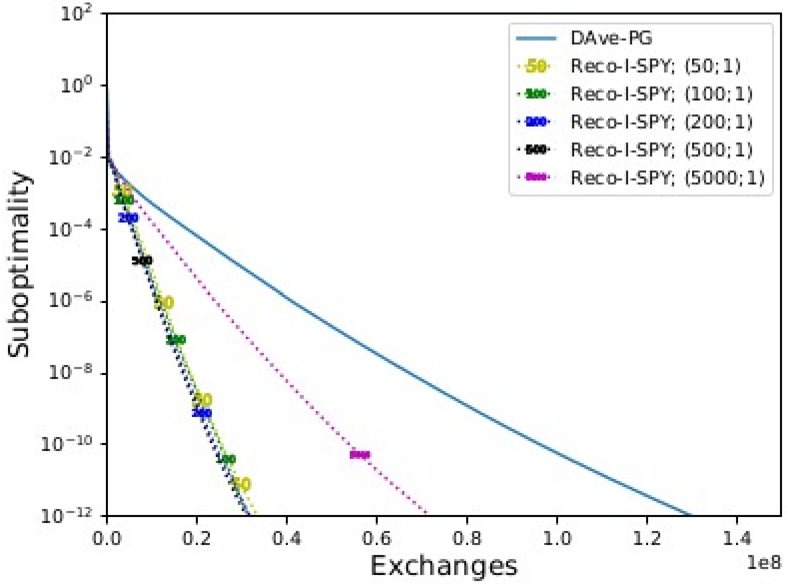


Lasso problem on synthetic data, $M = 10$ machines

$$\|Ax + b\|_2^2 + \lambda_1 \|x\|_1.$$

Experiments: What About Time?

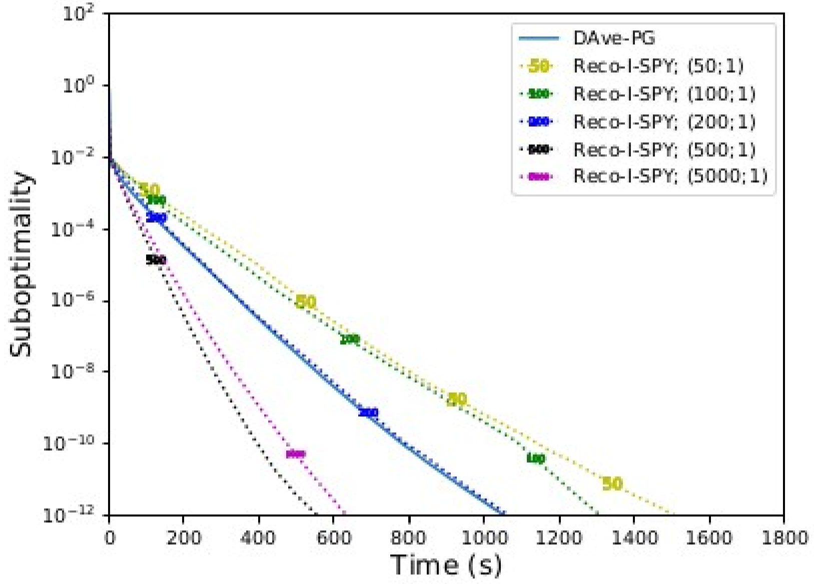


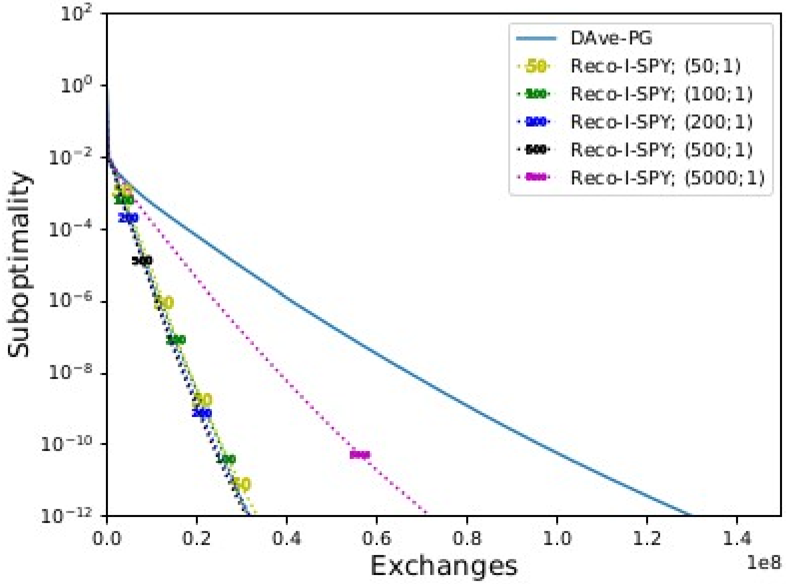


Logistic regression with elastic net regularizer on rcv1_train dataset ($n = 47236$ $m = 20242$) and $M = 10$ machines.

$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{m} \sum_{j=1}^{m} \log(1+\exp(-y_j z_j^\top x)) + \lambda_1 \|x\|_1 + \frac{\lambda_2}{2} \|x\|_2^2$$

Experiments: What About Time?





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Conclusion and Future Work

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+ Subspace descent algorithm for arbitrary regularized problem.

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→ Investigate (non)convex case.

→ Accelerated versions.

→ Combination with other sparsification techniques.


Thank You For
Your Attention!

Q & A

Practical for TV regularizer

Consider the set of artificial jumps $\mathcal{S} = \{n_1, n_2, \ldots, n_{l-1}\}$ and denote by $\mathcal{R} = \{i \notin \mathcal{S} : [\mathsf{S}_{\mathcal{M}}(x^k)]_i = 0\}$ the set of possible random entries. Fix the amount of sampled elements s and sample "first" element $\mathcal{R}_0$ uniformly in $\mathcal{R} = \{\mathcal{R}_i\}_{1 \le i \le r}$. Select "first s" elements starting from $\mathcal{R}_f$ considering the cyclic structure of the list of elements ($\mathcal{R}_{r+1} = \mathcal{R}_1$).

If l is small enough, it will not change the sparsity property of the random projection $P_{\mathfrak{S}^k}$; however, this modification will force all the projections to be block-diagonal with blocks' ends on positions $n_1, \ldots n_{l-1}$. In contrast with jumps(x^k) that we could not control, by adding l artificial jumps, we could guarantee that each block of the $P_{\mathfrak{S}^k}$ has at most $\lceil n/l \rceil$ rows. Since every random projection has end of the block on positions $\{n_i\}_{1 \le i \le l-1}$. P_ℓ also has such block structure and we could split the computation of Q_ℓ^{-1} and Q_ℓ into l independent parts and could be done in parallel.

Strategies for (A)RPSD

		(non-adaptive) subspace descent RPSD	adaptive subspace descent ARPSD
Subspace family		$\mathcal{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_c\}$	
Algorithm		$\begin{cases} y^k = \mathsf{Q}\left(x^k - \gamma \nabla f\left(x^k\right)\right) \\ z^k = P_{\mathfrak{S}^k}\left(y^k\right) + \left(I - P_{\mathfrak{S}^k}\right)\left(z^{k-1}\right) \\ x^{k+1} = \mathbf{prox}_{\gamma g}\left(\mathsf{Q}^{-1}\left(z^k\right)\right) \end{cases}$	
Selection	Option 1	$\mathcal{C}_i \in \mathfrak{S}^k$ with probability p	$\mathcal{C}_i \in \mathfrak{S}^k$ with probability $\begin{cases} p & \text{if } x^k \in \mathcal{M}_i \Leftrightarrow [\mathsf{S}_{\mathcal{M}}(x^k)]_i = 0 \\ 1 & \text{elsewhere} \end{cases}$
	Option 2	Sample s elements uniformly in $\mathcal{C}$	Sample s elements uniformly in $\{\mathcal{C}_i : x^k \in \mathcal{M}_i \text{ i.e. } [\mathsf{S}_{\mathcal{M}}(x^k)]_i = 0\}$ and add *all* elements in $\{\mathcal{C}_j : x^k \notin \mathcal{M}_j \text{ i.e. } [\mathsf{S}_{\mathcal{M}}(x^k)]_j = 1\}$

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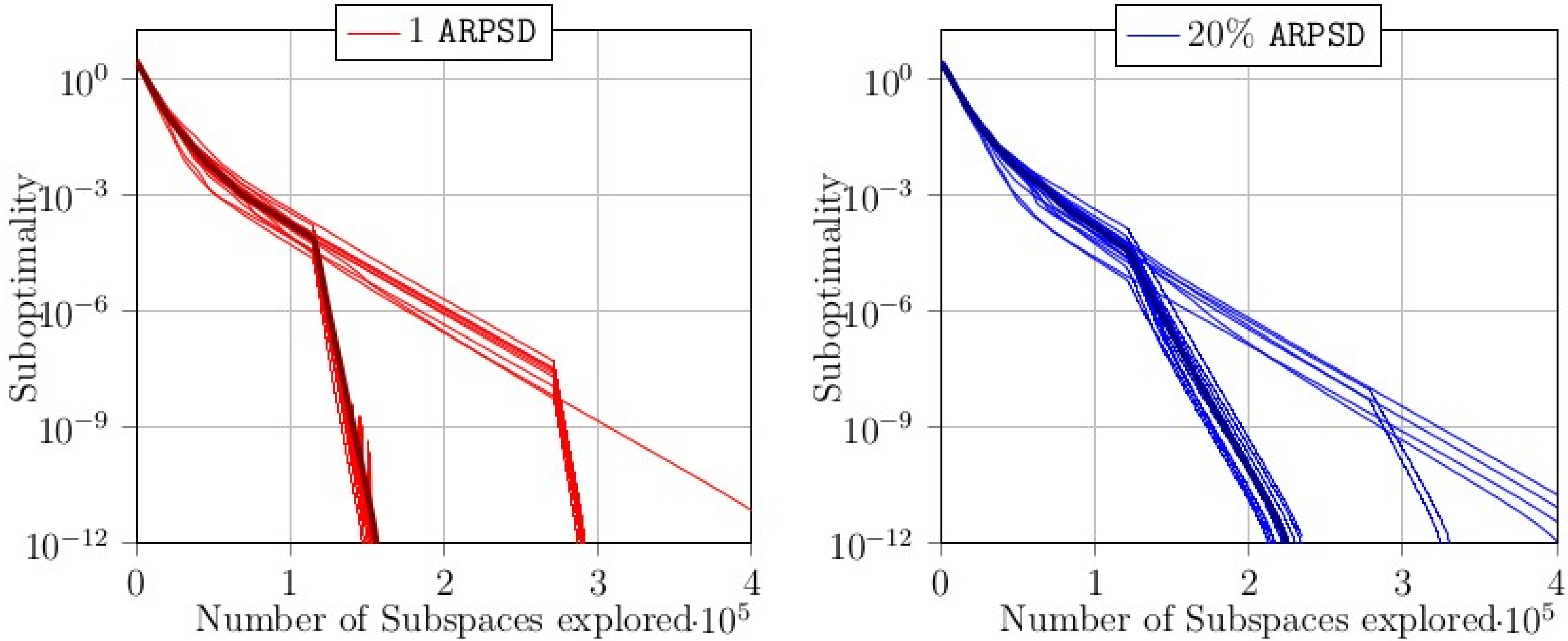
Practical robustness



Logistic regression with elastic net regularizer on rcv1_train dataset ($n = 47236$ $m = 20242$).

$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{m} \sum_{j=1}^{m} \log(1+\exp(-y_j z_j^\top x)) + \lambda_1 \|x\|_1 + \frac{\lambda_2}{2} \|x\|_2^2$$

Scaled SPY

Worker i

Initialize $x_i = x_i^+ = x = \bar{x}^0$

Calculate scaled probability vector $q = \left(\frac{p_{\min}}{p_1}, \frac{p_{\min}}{p_2}, \ldots, \frac{p_{\min}}{p_n}\right)$

while *not interrupted by master* **do**

  Receive x from master

  Draw sparsity mask $\mathbf{S}_p$ as

$$\mathbb{P}\left[j \in \mathbf{S}_p\right] = p_j$$

  $[x_i^+]_{\mathbf{S}_p} \leftarrow [q]_{\mathbf{S}_p} * [x - \gamma \nabla f_i(x)]_{\mathbf{S}_p} + [\mathbf{1^n} - q]_{\mathbf{S}_p} * [x_i]_{\mathbf{S}_p}$ [a]

  $\Delta \leftarrow x_i^+ - x_i$

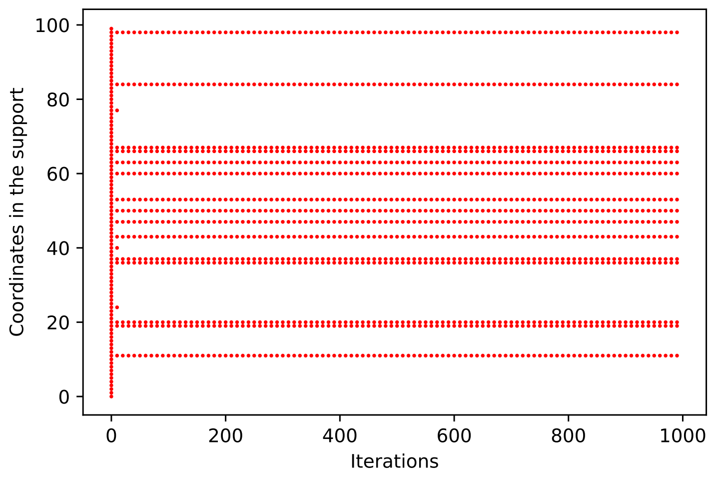
  Send $[\Delta]_{\mathbf{S}_p}$ to master

  $[x_i]_{\mathbf{S}_p} \leftarrow [x_i^+]_{\mathbf{S}_p}$

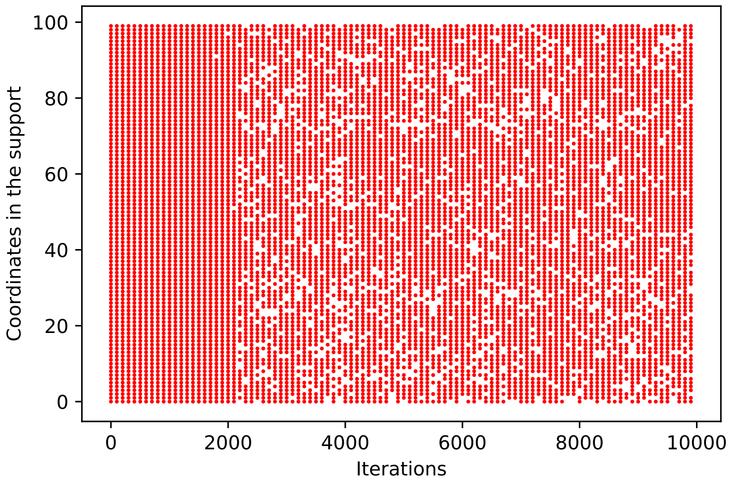
end

[a] Here we denote by $\mathbf{1^n} \in \mathbb{R}^n$ the identity vector and by $*$ we denote the coordinate-wise vector-to-vector multiplication.

Why not SGD



Prox GD



Prox SGD (minibatch of size 10)

Synthetic LASSO problem $\min \frac{1}{2}\|Ax - b\|_2^2 + \lambda_1 \|x\|_1$ for random generated matrix $A \in \mathbb{R}^{100\times 100}$ and vector $b \in \mathbb{R}^{100}$ and hyperparameter λ_1 chosen to reach 15% of density (amount of non-zero coordinates) of the final solution.

Non-degeneracy

Another way to define the non-degeneracy for the problem

$$\min_{x \in \mathbb{R}^n} f(x) + r(x)$$

is the following:

$$\nabla f(x^\star) \in \operatorname{ri} \partial r(x^\star).$$

In case of ℓ_1 regularizer $r(x) = \lambda_1 \|x\|_1$ this can be written explicitly as

$$\left| \nabla f(x^\star)_{[j]} \right| < \lambda_1 \qquad \text{for all } j \in \operatorname{supp}(x^\star).$$

C2 and C3

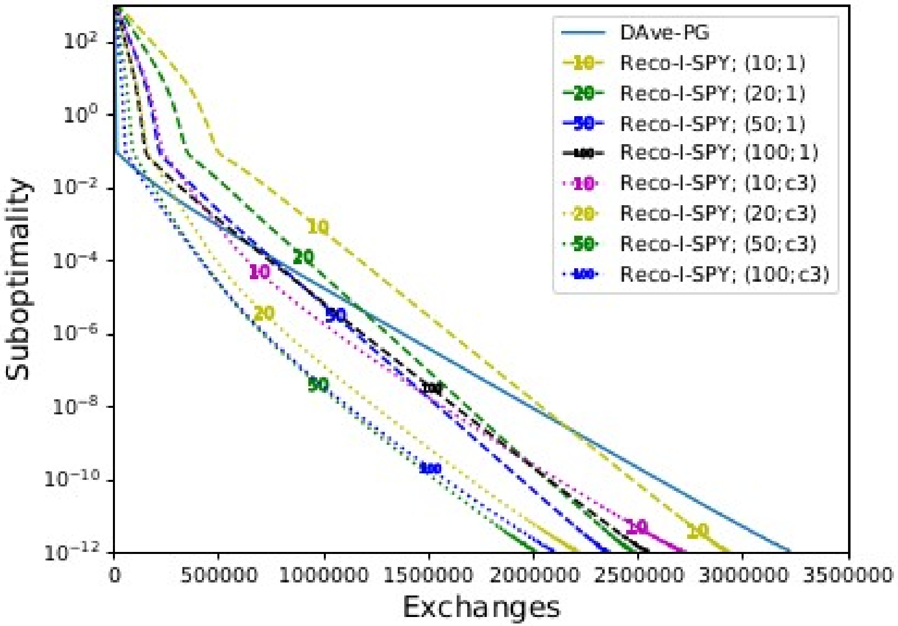
C_2 (absolute accuracy): Run `I-SPY` until it finds $x_{\ell+1}$ such that

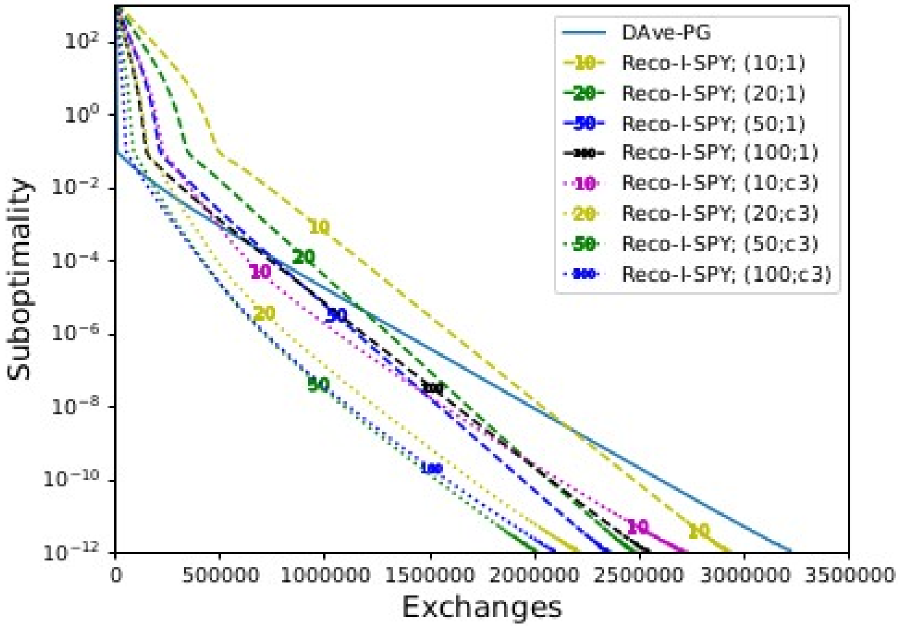
$$\|x_{\ell+1} - \mathbf{prox}_{F/\rho}(x_\ell)\|_2^2 \leq \frac{(1-\delta)\rho}{(2\mu+\rho)\ell^{1+\delta}} \|x_\ell - \mathbf{prox}_{F/\rho}(x_\ell)\|_2^2.$$

C_3 (relative accuracy): Run `I-SPY` until it finds $x_{\ell+1}$ such that

$$\|x_{\ell+1} - \mathbf{prox}_{F/\rho}(x_\ell)\|_2^2 \leq \frac{\rho}{4(2\mu+\rho)\ell^{2+2\delta}} \|x_{\ell+1} - x_\ell\|_2^2.$$

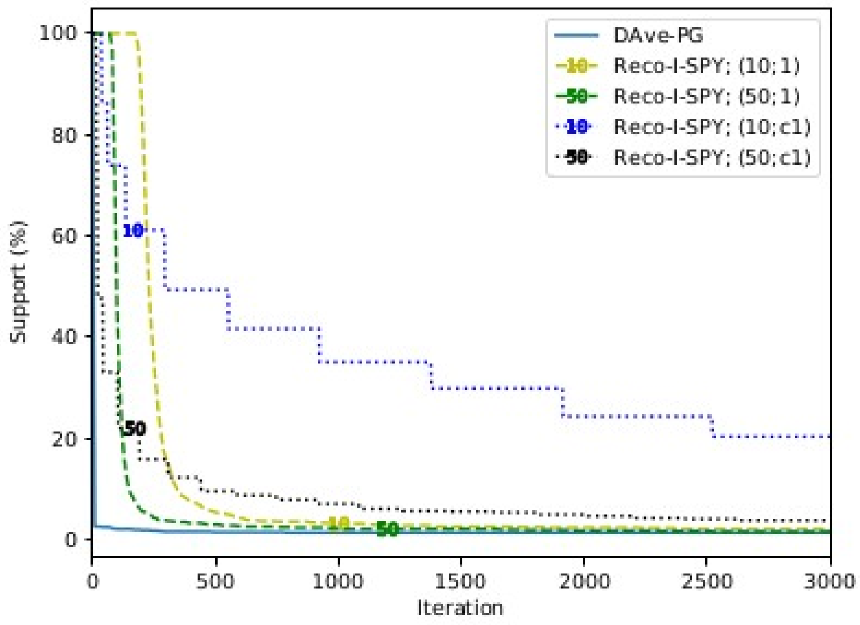
1 epoch Vs C3 (Exps)

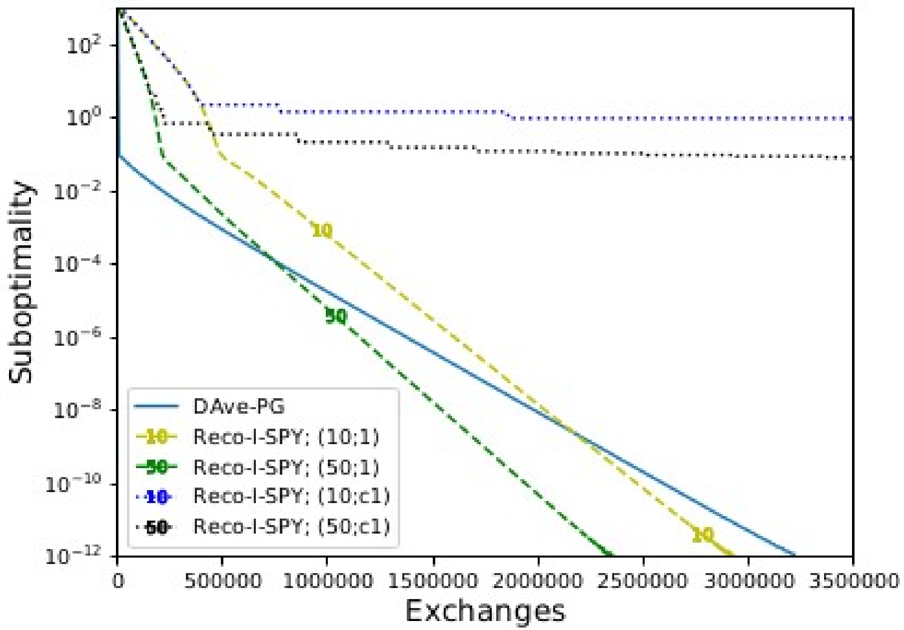




Synthetic LASSO problem min $\frac{1}{2}\|Ax - b\|_2^2 + \lambda_1\|x\|_1$ for random generated matrix $A \in \mathbb{R}^{10000\times1000}$ and vector $b \in \mathbb{R}^{10000}$ and hyperparameter λ_1 chosen to reach 1% of density (amount of non-zero coordinates) of the final solution.

1 epoch Vs C1 (Exps)





Synthetic LASSO problem min $\frac{1}{2}\|Ax - b\|_2^2 + \lambda_1\|x\|_1$ for random generated matrix $A \in \mathbb{R}^{10000\times1000}$ and vector $b \in \mathbb{R}^{10000}$ and hyperparameter λ_1 chosen to reach 1% of density (amount of non-zero coordinates) of the final solution.